5B.3 The Relaxation Spectrum [OH]

Inserting (5B.3-1) into (5.3-4 & 5):

$$\eta'(\omega) = \int_0^\infty \frac{H(\lambda)}{\lambda}\left\{\int_0^\infty e^{-s/\lambda}\cos\omega s\, ds\right\}d\lambda$$

$$= \int_0^\infty \frac{H(\lambda)}{\lambda}\left\{\frac{\lambda}{1+(\lambda\omega)^2}\right\}d\lambda = \int_0^\infty \frac{H(\lambda)\,d\lambda}{1+(\lambda\omega)^2}$$

$$\frac{\eta''(\omega)}{\omega} = \int_0^\infty \frac{H(\lambda)}{\lambda\omega}\left\{\int_0^\infty e^{-s/\lambda}\sin(\omega s)\,ds\right\}d\lambda$$

$$= \frac{1}{\omega}\int_0^\infty \frac{H(\lambda)}{\lambda}\left\{\frac{\lambda^2\omega}{1+(\lambda\omega)^2}\right\}d\lambda = \int_0^\infty \frac{\lambda H(\lambda)\,d\lambda}{1+(\lambda\omega)^2}$$

For the generalized Maxwell Model:

$$\eta'(\omega) = \sum_{k=1}^\infty \frac{\eta_k}{1+(\lambda_k\omega)^2} \; ; \; \frac{\eta''}{\omega} = \sum_{k=1}^\infty \frac{\eta_k\lambda_k}{1+(\lambda_k\omega)^2}$$

This is a special case: $H(\lambda) = \sum_{k=1}^\infty \eta_k\,\delta(\lambda-\lambda_k)$.

That is $H(\lambda)$ is a "continuum" generalization of η_k & λ_k.

and consider $\eta_k = H(\lambda_k)\,\Delta\lambda_k \Rightarrow$ Maxwell model is Rieman sum of Integral

5B.3 The Relaxation Spectrum

From eqn. (5B.3-1), the relaxation modulus can be written in the form

$$G(s) = \int_0^\infty \frac{H(\lambda)}{\lambda} e^{-s/\lambda} \, d\lambda = \int_{-\infty}^{\infty} H(\lambda) e^{-s/\lambda} \, d(\ln \lambda) \quad \text{—(1)}$$

From eqns. (5.3-4) and (5.3-5)

$$\eta'(\omega) = \int_0^\infty G(s) \cos(\omega s) \, ds \quad \text{—(2)}$$

$$\eta''(\omega) = \int_0^\infty G(s) \sin(\omega s) \, ds \quad \text{—(3)}$$

Substituting for $G(s)$ from (1),

$$\eta'(\omega) = \int_0^\infty \frac{H(\lambda)}{\lambda} \left\{ \int_0^\infty e^{-s/\lambda} \cos(\omega s) \, ds \right\} d\lambda \quad \text{—(4)}$$

$$\eta''(\omega) = \int_0^\infty \frac{H(\lambda)}{\lambda} \left\{ \int_0^\infty e^{-s/\lambda} \sin(\omega s) \, ds \right\} d\lambda \quad \text{—(5)}$$

Simplifying eqns. (4) and (5) we get

$$\eta'(\omega) = \int_0^\infty \frac{H(\lambda)}{\lambda} \left\{ \frac{\lambda}{1+\lambda^2\omega^2} \right\} d\lambda = \int_0^\infty \frac{H(\lambda) \, d\lambda}{1 + (\lambda\omega)^2} \quad \text{—(6)}$$

$$\frac{\eta''(\omega)}{\omega} = \frac{1}{\omega} \int_0^\infty \frac{H(\lambda)}{\lambda} \left\{ \frac{\lambda(\lambda\omega)}{1+(\lambda\omega)^2} \right\} d\lambda = \int_0^\infty \frac{\lambda \, H(\lambda) \, d\lambda}{1 + (\lambda\omega)^2} \quad \text{—(7)}$$

For the generalized Maxwell model (eqns 5.3-8 and -9),

$$\eta'(\omega) = \sum_{k=1}^{\infty} \frac{\eta_k}{1+(\lambda_k\omega)^2} \quad ; \quad \frac{\eta''}{\omega} = \sum_{k=1}^{\infty} \frac{\eta_k \lambda_k}{1+(\lambda_k\omega)^2}$$

Comparing with the above expressions, we see $H(\lambda) = \eta_k$

Let $\frac{k^2}{\lambda\omega} = t$

$$\frac{\eta''}{\eta_0} \cong \frac{6\lambda\omega}{\pi^2}\left(\frac{1}{(\lambda\omega)^2}\int_0^\infty \frac{dk}{\left(\frac{k^2}{\lambda\omega}\right)^2+1} - \frac{1}{2(\lambda\omega)^2}\right) = \frac{6}{\pi^2 \lambda\omega}\left(\int_0^\infty \frac{1}{t^2+1}\,\frac{\sqrt{\lambda\omega}}{2}\,\frac{dt}{\sqrt{t}} - \frac{1}{2}\right)$$

$$\cong \frac{6\sqrt{\lambda\omega}}{2\pi^2(\lambda\omega)}\left(\int_0^\infty \frac{dt}{(t^2+1)\sqrt{t}} - \frac{1}{\sqrt{\lambda\omega}}\right); \quad \text{Let } t = \tan\theta, \; d\theta = \frac{dt}{1+t^2} \Rightarrow \frac{\eta''}{\eta_0} \simeq \frac{6\sqrt{\lambda\omega}}{2\pi^2\lambda\omega}\left(\int_0^{\pi/2} \tan^{-1/2}\theta\, d\theta - \right.$$

$$\simeq \frac{3}{\pi^2}\left(\frac{\sqrt{\lambda\omega}}{(\lambda\omega)}\left[\frac{\pi}{2\cos\frac{\pi}{4}}\right] - \frac{1}{\lambda\omega}\right) = \frac{3}{\pi^2}\left[\frac{\sqrt{2}\pi}{2}\frac{1}{\sqrt{\lambda\omega}} - \frac{1}{\lambda\omega}\right]$$

SB.3

24. a) $\eta^* = \int_0^\infty G(s)\, e^{-i\omega s}\, ds \; ; \quad G(s) = \int_0^\infty \frac{H(\lambda)}{\lambda} e^{-s/\lambda}\, d\lambda$

$$= \int_0^\infty\int_0^\infty \frac{H(\lambda)}{\lambda} e^{-s(\frac{1}{\lambda}+i\omega)}\, ds\, d\lambda = \int_0^\infty\int_0^\infty \frac{H(\lambda)}{\lambda} e^{-s(\frac{1}{\lambda}+i\omega)}\, d\lambda\, ds$$

$$= \int_0^\infty \frac{H(\lambda)}{-\lambda(\frac{1}{\lambda}+i\omega)} e^{-s(\frac{1}{\lambda}+i\omega)}\Big|_0^\infty d\lambda = \int_0^\infty \frac{H(\lambda)}{1+i\omega\lambda}\, d\lambda$$

$$= \int_0^\infty \frac{H(\lambda)(1-i\omega\lambda)}{1+(\omega\lambda)^2}\, d\lambda = \int_0^\infty \frac{H(\lambda)}{1+(\lambda\omega)^2}\, d\lambda - i\int_0^\infty \frac{\omega\lambda H(\lambda)}{1+(\omega\lambda)^2}\, d\lambda$$

$$\eta^* = \eta' - i\eta'' \Rightarrow \eta' = \int_0^\infty \frac{H(\lambda)}{1+(\lambda\omega)^2}\, d\lambda, \quad \eta'' = \int_0^\infty \frac{\omega\lambda H(\lambda)}{1+(\omega\lambda)^2}\, d\lambda$$

b) For Maxwell Model $\eta^*_m = \frac{1-i\omega\lambda_0}{1+(\omega\lambda_0)^2}\eta_0$

Since $\eta^* = \int_0^\infty \frac{1-i\omega\lambda}{1+(\omega\lambda)^2} H(\lambda)\, d\lambda$ we choose $\boxed{H(\lambda) = \eta_0\, \delta(\lambda-\lambda_0)}$

$\lambda_0 > 0$

$$\eta^*_m = \int_0^\infty \frac{1-i\omega\lambda}{1+(\omega\lambda)^2}\eta_0\, \delta(\lambda-\lambda_0)\, d\lambda$$

$$= \frac{1-i\omega\lambda_0}{1+(\omega\lambda_0)^2}\eta_0$$

5B.5 High-Frequency Expressions for η' and η'' for the Generalized Maxwell Model [JDS]

a) This is a special case of b) using $\zeta(2) = \frac{\pi^2}{6}$

b) $$\sum_{k=1}^{\infty} \frac{k^{\alpha}}{k^{2\alpha}+(\lambda\omega)^2} = \sum_{k=0}^{\infty} \frac{k^{\alpha}}{k^{2\alpha}+(\lambda\omega)^2}$$

$$\cong \int_0^{\infty} \frac{k^{\alpha} d\alpha}{k^{2\alpha}+(\lambda\omega)^2} \quad ; \quad \text{change: } t = k^{\alpha}/\lambda\omega$$

$$\cong \frac{(\lambda\omega)^{\frac{1}{\alpha}-1}}{\alpha} \int_0^{\infty} \frac{t^{1/\alpha} dt}{t^2+1}$$

$$\cong \frac{\pi (\lambda\omega)^{\frac{1}{\alpha}-1}}{(\alpha+1)\sin[\pi(\alpha+1)/2\alpha]}$$

Thus: $$\frac{\eta'}{\eta_0} \cong \frac{1}{\zeta(\alpha)} \left\{ \frac{\pi (\lambda\omega)^{\frac{1}{\alpha}-1}}{(1+\alpha)\sin[\pi(1+\alpha)/2\alpha]} \right\} \quad (5.3\text{-}14);$$

Likewise:
$$\sum_{k=1}^{\infty} \frac{1}{k^{2\alpha}+(\lambda\omega)^2} = \sum_{k=0}^{\infty} \frac{1}{k^{2\alpha}+(\lambda\omega)^2} - \frac{1}{(\lambda\omega)^2}$$

$$\sum_{k=0}^{\infty} \frac{1}{k^{2\alpha}+(\lambda\omega)^2} \cong \int_0^\infty \frac{dk}{k^{2\alpha}+(\lambda\omega)^2} + \frac{1}{2(\lambda\omega)^2} \quad ; \quad t := \frac{k^\alpha}{\lambda\omega}$$

$$\cong \frac{(\lambda\omega)^{\frac{1}{\alpha}-2}}{\alpha} \underbrace{\int_0^\infty \frac{t^{1-1/\alpha}}{t^2+1}}_{\pi/2\sin[\pi/2\alpha]} + \frac{1}{2(\lambda\omega)^2}$$

Thus:

$$\frac{\eta''}{\eta_0} \cong \frac{1}{2\lambda\omega\zeta(\alpha)} \left\{ \frac{(\lambda\omega)^{1/\alpha}}{\alpha \sin[\pi/2\alpha]} - 1 \right\} \quad (5.3\text{-}15)$$

where Eqs. (5.3-10 & 11) were used.

$\boxed{5B.5}$ a) This is just a special case of b). However, you do need

$\zeta(2) = \pi^2/6$ by (23.2.24) of Abramowitz & Stegun ; Therefore, I'll only do b).

$$\text{b)} \quad \sum_{k=1}^{\infty} \frac{k^\alpha}{k^{2\alpha}+(\lambda\omega)^2} = \sum_{k=0}^{\infty} \frac{k^\alpha}{k^{2\alpha}+(\lambda\omega)^2} \cong \int_0^\infty \frac{k^\alpha \, d\alpha}{k^{2\alpha}+(\lambda\omega)^2} \quad ; \left.\begin{array}{l} t = k^\alpha/\lambda\omega \\ k = \sqrt[\alpha]{\lambda\omega t} \end{array}\right\} \text{change of var}$$

$$\cong \int_0^\infty \frac{\lambda\omega t}{(\lambda\omega t)^2 + (\lambda\omega)^2} \cdot \frac{(\lambda\omega)^{1/\alpha} dt}{\alpha \, t^{1-1/\alpha}} = \frac{(\lambda\omega)^{1/\alpha - 1}}{\alpha} \int_0^\infty \frac{t^{1/\alpha} dt}{t^2+1}$$

$$\cong \frac{\pi (\lambda\omega)^{1/\alpha - 1}}{(\alpha+1) \sin[\pi(\alpha+1)/2\alpha]} \quad \text{by 3.241-2 of Gradshteyn \& Ryzhik}$$

$$\text{Thus,} \quad \frac{\eta'}{\eta_0} \cong \frac{1}{\zeta(\alpha)} \left\{ \frac{\pi (\lambda\omega)^{1/\alpha - 1}}{(1+\alpha) \sin[\pi(1+\alpha)/(2\alpha)]} \right\}$$

$$\text{Also,} \quad \sum_{k=1}^{\infty} \frac{1}{k^{2\alpha}+(\lambda\omega)^2} = \sum_{k=0}^{\infty} \frac{1}{k^{2\alpha}+(\lambda\omega)^2} - \frac{1}{(\lambda\omega)^2}$$

$$\sum_{k=0}^{\infty} \frac{1}{k^{2\alpha}+(\lambda\omega)^2} \cong \int_0^\infty \frac{dk}{k^{2\alpha}+(\lambda\omega)^2} + \frac{1}{2(\lambda\omega)^2} \quad ; \; t = \frac{k^\alpha}{\lambda\omega} \text{ as before}$$

$$\int^\infty \frac{dk}{k^{2\alpha}+(\lambda\omega)^2} = \frac{(\lambda\omega)^{1/\alpha - 2}}{\alpha} \int_0^\infty \frac{t^{1-1/\alpha} dt}{t^2+1} = \frac{\pi (\lambda\omega)^{1/\alpha - 2}}{2\alpha \sin[\pi/2\alpha]}$$

$$\therefore \quad \frac{\eta''}{\eta_0} \cong \frac{1}{2\lambda\omega\zeta(\alpha)} \left\{ \frac{(\lambda\omega)^{1/\alpha}}{\alpha \sin[\pi/2\alpha]} - 1 \right\} //$$

5B.8 Linear Viscoelasticity from the Doi-Edwards Kinetic Theory for Polymer Melts [RBB]

Start with Eq. 5.2-14 with sum on odd indices:

$$G(s) = \sum_{k=odd} \frac{\eta_k}{\lambda_k} e^{-s/\lambda_k}$$

Now substitute the expressions in Eq. 5B.8-2

$$G(s) = \sum_{k,odd} \frac{\eta_0 \lambda_k}{\sum \lambda_k^2} e^{-s/\lambda_k}$$

$$= \eta_0 \sum_{k,odd} \frac{(\lambda/\pi^2 k^2)}{\left(\frac{\lambda^2}{\pi^4} \sum_{odd} \frac{1}{k^4}\right)} e^{-s\pi^2 k^2/\lambda}$$

Since $\sum_{k=odd} \frac{1}{k^4} = \frac{\pi^4}{96}$ (see # 48.14 in Dwight)

we get

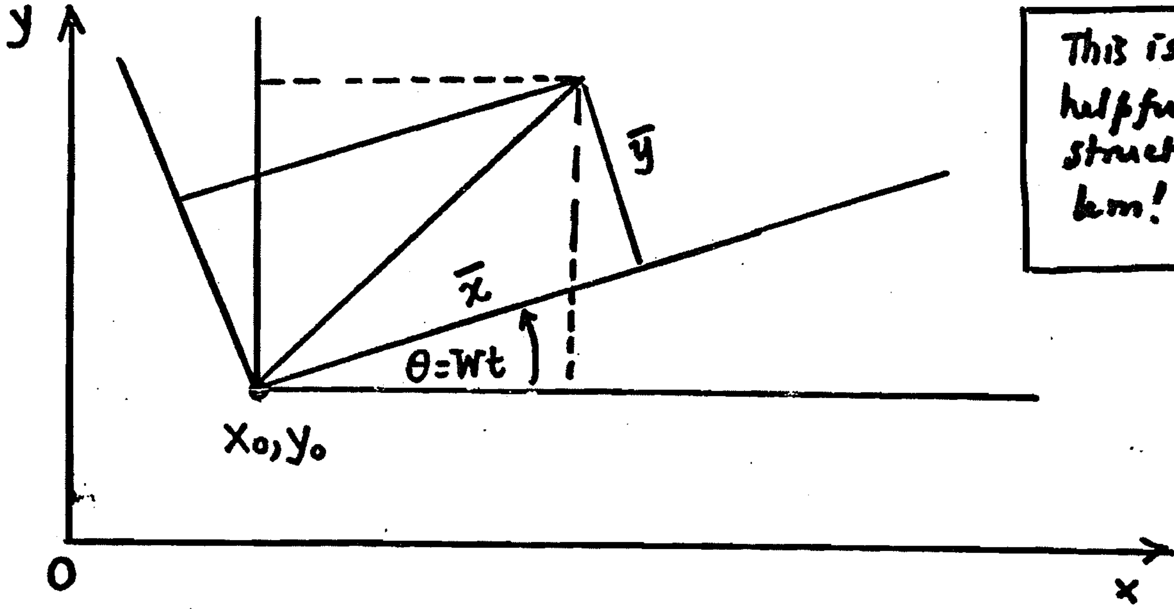
$$G(s) = \frac{\eta_0 \pi^2/\lambda}{\pi^4/96} \sum_{k,odd} \frac{1}{k^2} e^{-\pi^2 k^2 s/\lambda}$$

or

$$G(s) = \frac{96\,\eta_0}{\pi^2 \lambda} \sum_{k,odd} \frac{1}{k^2} e^{-\pi^2 k^2 s/\lambda}$$

Note: In the first printing the factor $1/k^2$ was omitted

5C.2 Displacement Gradients in the Turntable Problem [RBB]



This is a very helpful + instructive problem! rb

Relations between coordinates: (here $C = \cos Wt$, $S = \sin Wt$)

$$\begin{cases} (x - x_o) = \bar{x}C - \bar{y}S \\ (y - y_o) = \bar{x}S + \bar{y}C \end{cases} \quad \text{or} \quad \begin{cases} \bar{x} = (x - x_o)C + (y - y_o)S & (5.5\text{-}1) \\ \bar{y} = -(x - x_o)S + (y - y_o)C & (5.5\text{-}2) \end{cases}$$

a. Now in the rotating coordinate frame $v_{\bar{x}} = \dot{\gamma}\bar{y}$ so that

$$\bar{x}' - \bar{x} = \dot{\gamma}\bar{y}\,(t' - t) \quad \text{and} \quad \bar{y}' - \bar{y} = 0 \qquad (A)$$

Next the displacement $u_x = x' - x = (x' - x_o) - (x - x_o)$ is obtained from the relations among the coordinates:

$$u_x = (\bar{x}'C' - \bar{y}'S') - (x - x_o)$$

$$= \left[\bar{x} + \dot{\gamma}\bar{y}\,(t' - t)\right]C' - \bar{y}\,S' - (x - x_o) \quad \text{using (A)}$$

Next, eliminate $\bar{x}$ and $\bar{y}$ using Eqs. 5.5-1 and 5.5-2:

$$u_x = (x-x_0)[C'C + S'S - \dot{\gamma}(t'-t)C'S] + (y-y_0)[C'S - S'C + \dot{\gamma}(t'-t)C'C] - (x-x_0)$$

And similarly

$$u_y = (x-x_0)[S'C - C'S - \dot{\gamma}(t'-t)S'S] + (y-y_0)[S'S + C'C + \dot{\gamma}(t'-t)S'C] - (y-y_0)$$

These last two equations are equivalent to Eqs. 5C.2-1 and 2.

b. Next get the displacement gradients:

$$\begin{cases} \frac{\partial}{\partial x}u_x = C'C + S'S - \dot{\gamma}(t'-t)C'S - 1 \\ \frac{\partial}{\partial x}u_y = S'C - C'S - \dot{\gamma}(t'-t)S'S \\ \frac{\partial}{\partial y}u_x = C'S - S'C + \dot{\gamma}(t'-t)C'C \\ \frac{\partial}{\partial y}u_y = S'S + C'C + \dot{\gamma}(t'-t)S'C - 1 \end{cases}$$

Then $\underline{\underline{\gamma}} = \nabla\underline{u} + (\nabla\underline{u})^{\dagger}$. The yx-component is:

$$\gamma_{yx}(t,t') = \dot{\gamma}(t'-t)(C'C - S'S)$$
$$= \dot{\gamma}(t'-t)\cos 2W(t'+t)$$

Note that THE COMPONENTS OF $\nabla\underline{u}$ are <u>NOT SMALL</u> for non-vanishing W !

C. From Eq. 5.2-19:

$$\tau_{yx} = \int_{-\infty}^{t} M(t-t')\, \dot\gamma\,(t'-t)\, \cos 2W(t'+t)\, dt'$$

$$= -\int_0^\infty M(s)\, s \cos 2W(2t-s)\, ds \cdot \dot\gamma \quad (5C.2\text{-}7)$$

$$= +\dot\gamma \int_0^\infty \frac{\partial G}{\partial s}\, s \cos 2W(2t-s)\, ds$$

$$= \dot\gamma\, G\, s \cos 2W(2t-s) \Big|_0^\infty$$

Note $G(s) \to 0$ exp'ly. as $s \to \infty$

$$-\dot\gamma \int_0^\infty G(s) \left[\cos 2W(2t-s) + 2Ws \sin 2W(2t-s)\right] ds$$

Hence $$\tau_{yx}(t=0) = -\dot\gamma \int_0^\infty G(s) \left[\cos 2Ws \underbrace{- 2Ws \sin 2Ws}_{\text{This term is not in Eq. 5.5-7.}}\right] ds$$

NOTE that Eq. 5.2-19 was obtained from Eq. 5.2-18 by using Eq. 5.2-7: $\partial \underline{\underline{\gamma}}/\partial t' = \underline{\underline{\dot\gamma}}(t')$. Just after Eq. 5.2-7 it is stated that the components of $\nabla \underline{u}$ must be <u>infinitesimally</u> <u>small</u> in order for Eq. 5.2-7 to hold. But it was shown in (b) that the components of $\nabla \underline{u}$ are <u>not</u> <u>small</u>. Therefore there is no reason why the above expression for $\tau_{yx}(t=0)$ should agree with Eq. 5.5-7.

d. When $W \to 0$, $C = 1 + \cdots$, $C' = 1 + \cdots$, $S = Wt + \cdots$, and $S' = Wt' + \cdots$. Then

$$(\nabla \underline{u})_{xx} = 1 + W^2 t t' - \dot{\gamma}(t'-t) Wt - 1 + \cdots = \dot{\gamma}(t-t') Wt + \cdots$$

WHY IS $(\nabla \underline{u})_{yx}$ small? $(t-t')$ can go to ∞!

$$(\nabla \underline{u})_{xy} = Wt' - Wt - \dot{\gamma}(t'-t) W^2 t t' + \cdots = -W(t-t') + \cdots$$

$$(\nabla \underline{u})_{yx} = Wt - Wt' + \dot{\gamma}(t'-t) + \cdots = W(t-t') + \cdots - \dot{\gamma}(t-t')$$

$$(\nabla \underline{u})_{yy} = W^2 t t' + 1 + \dot{\gamma}(t'-t) Wt' - 1 + \cdots = -\dot{\gamma}(t-t') Wt' + \cdots$$

In the limit as $W \to 0$, using (c), we get from Eq. 5.5-7

$$\tau_{yx}(t=0) = -\dot{\gamma} \int_0^\infty G(s) \, (1 + \cdots) \, ds$$

goes to zero at $W \to 0$

or $\quad \eta_0 = \int_0^\infty G(s) \, ds$

From Eq. 5C.2-7, at $t=0$,

$$\tau_{yx}(t=0) = -\dot{\gamma} \int_0^\infty M(s) (\cos Ws) \, s \, ds$$

Then as $W \to 0$, we get

goes to zero as $W \to 0$

$$\tau_{yx}(t=0) = -\dot{\gamma} \int_0^\infty M(s) \, s \, (1 + \cdots) \, ds$$

$$= -\dot{\gamma} \left[\int_0^\infty -\frac{dG}{ds} \, s \, ds \right]$$

$$= +\dot{\gamma} \left[+sG \Big|_0^\infty - \int_0^\infty G \, ds \right]$$

whence $\quad \eta_0 = \int_0^\infty G(s) \, ds$

e. From (b) we get the components of $\underline{\underline{\gamma}} = \nabla\underline{u} + (\nabla\underline{u})^{\dagger}$:

$$\gamma_{xx} = 2\left[C'C + S'S - \dot{\gamma}(t'-t)\,C'S - 1\right]$$

$$\gamma_{xy} = \gamma_{yx} = \dot{\gamma}(t'-t)(S'S + C'C)$$

$$\gamma_{yy} = 2\left[S'S + C'C + \dot{\gamma}(t'-t)\,S'C - 1\right]$$

Then

$$\frac{\partial}{\partial t'}\gamma_{xx} = 2\left[-S'C + C'S + \dot{\gamma}(t'-t)\,S'S\right]W - 2\dot{\gamma}C'S$$

$$\frac{\partial}{\partial t'}\gamma_{xy} = \frac{\partial}{\partial t'}\gamma_{yx} = \dot{\gamma}\left(S'S + C'C\right) + \dot{\gamma}(t'-t)(C'S - S'C)W$$

$$\frac{\partial}{\partial t'}\gamma_{yy} = 2\left[C'S - S'C + \dot{\gamma}(t'-t)\,C'C\right]W + 2\dot{\gamma}S'C$$

Hence as $W \to 0$ we get only contributions from term:

$$\frac{\partial}{\partial t'}\underline{\underline{\gamma}}(t,t') = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\dot{\gamma}$$

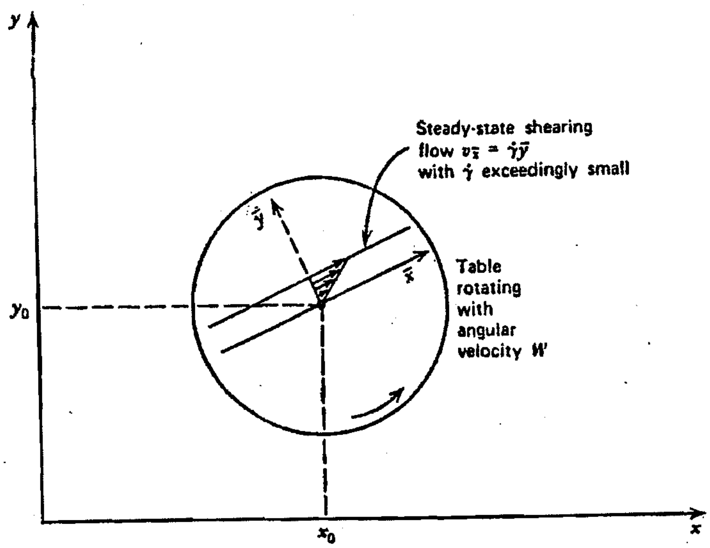
From Eq. 5.5-5, we get for $W \to 0$

$$\dot{\underline{\underline{\gamma}}}(t,t') = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}\dot{\gamma}$$

Is $(\nabla\underline{u})_{yx}$ always small?

Hence Eq. 5.2-7 is true only as $W \to 0$. According to (d), the components of $\nabla\underline{u}$ are vanishingly small for $W \to 0$.

5C.2 Displacement Gradients in the Turntable Problem



a.

x, y : Location of fluid particle at time t.
x', y' : Location of same particle at time t'.
$\bar{x}, \bar{y}$: Location of particle with respect to turntable axis.

$$\bar{x}' - \bar{x} = \dot{\gamma}\,\bar{y}\,(t'-t) \qquad (1)$$
$$\bar{y}' = \bar{y}$$

$$\bar{x} = (x-x_0)C + (y-y_0)S \;;\; \bar{x}' = (x'-x_0)C' + (y'-y_0)S$$
$$\bar{y} = -(x-x_0)S + (y-y_0)C \;;\; \bar{y}' = -(x'-x_0)S' + (y'-y_0)C' \qquad (2)$$

(See Eqns. 5.5-1 & 2, DPL).

$$\therefore (x'-x_0)C' + (y'-y_0)S - (x-x_0)C - (y-y_0)S = \dot{\gamma}(t'-t) * \left[(y-y_0)C - (x-x_0)S\right]$$

$$(y'-y_0)C' - (x'-x_0)S' = (y-y_0)C - (x-x_0)S$$

Simplifying, we obtain the required results, i.e.

5. C-2 (Contd.)

$$x' - x_o = (x - x_0)\left[CC' + SS' - \dot{\gamma}(t'-t)\,C'S'\right] + (y - y_0)\left[C'S - S'C + \dot{\gamma}(t'-t)\,CC'\right]$$

$$y' - y_o = (x - x_0)\left[S'C - C'S - \dot{\gamma}(t'-t)\,S'S\right] + (y - y_0)\left[S'S + C'C + \dot{\gamma}(t'-t)\,S'C'\right] \qquad (3)$$

b.

$$(\nabla u)_{xx} = \frac{\partial u_x}{\partial x} = \frac{\partial x'}{\partial x} - 1 = C'C + S'S - \dot{\gamma}(t'-t)\,C'S - 1$$

$$(\nabla u)_{xy} = \frac{\partial u_y}{\partial x} = \frac{\partial y'}{\partial x} = S'C - C'S - \dot{\gamma}(t'-t)\,S'S$$

$$(\nabla u)_{yx} = \frac{\partial u_x}{\partial y} = \frac{\partial x'}{\partial y} = C'S - S'C + \dot{\gamma}(t'-t)\,C'C$$

$$(\nabla u)_{yy} = \frac{\partial u_y}{\partial y} = \frac{\partial y'}{\partial y} - 1 = S'S + C'C + \dot{\gamma}(t'-t)\,S'C - 1$$

The infinitesimal strain tensor, γ is defined as

$$\underline{\underline{\gamma}} = \left\{(\nabla \underline{u}) + (\nabla \underline{u})^{\dagger}\right\}$$

$$\therefore \quad \underline{\underline{\gamma}} = \begin{bmatrix} 2(\nabla u)_{xx} & (\nabla u)_{yx} + (\nabla u)_{xy} & 0 \\ (\nabla u)_{xy} + (\nabla u)_{yx} & 2(\nabla u)_{yy} & 0 \\ 0 & 0 & 0 \end{bmatrix} \qquad (4)$$

5C.2 (Contd.)

c. From eqn. 5.2-19,

$$\tau_{yx} = \int_{-\infty}^{t} M(t-t')\, \gamma_{yx}(t,t')\, dt'$$

Substituting from part (b),

$$\tau_{yx} = \int_{-\infty}^{t} M(t-t')\left[C'S - S'C + \dot{\gamma}(t'-t)(CC'-SS') + S'C - C'S\right] dt'$$

$$\therefore\ \tau_{yx} = -\int_{0}^{\infty} M(s)\, \dot{\gamma}\, s\, \cos[W(2t-s)]\, ds \qquad ; \quad s = t-t'$$

Integrating the above eqn. by parts, we get

$$\tau_{yx} = -\left[-s\dot{\gamma}\cos[W(2t-s)]\, G(s)\Big|_0^{\infty} - \int_0^{\infty} G(s)\,\dot{\gamma}\left(\cos[W(2t-s)] + Ws\,\sin[W(2t-s)]\right) ds \right]$$

$$\tau_{yx} = -\int_0^{\infty} G(s)\,\dot{\gamma}\left(\cos[W(2t-s)] + Ws\,\sin[W(2t-s)]\right) ds \qquad -(5)$$

The expression for the shear stress given by eqn. (5) is different from that given by eqn. (5.5-7), for arbitrary W. However, one should keep in mind, that eqns. (5.2-18 & 19) are valid only in the limit of infinitesimally small displacement gradients.