

## 13.024 Problem Set 10:

Due: May 6, 2003

### Numerical integration of 1<sup>st</sup> order ODE

1. A spherical buoy of mass  $m_0$  and radius  $R$  is anchored to the bottom of the ocean and is held in a fully submerged position. At time  $t_0=0$  the anchor chain fitting breaks, allowing the buoy to break free, while at the same time springing a leak into the sphere of constant mass flow rate  $Q$ . As the buoy rises or falls, the forces on it are the (buoyancy –the weight, which is upward if positive and downward if negative), and a viscous drag force opposite to the direction of motion. The magnitude of this force is  $\frac{1}{2} \rho V |V| (\pi R^2) C_d$ .  $\rho$  is the fluid density,  $V$  is the velocity of the buoy and  $C_d$  is the drag coefficient of the buoy.

Give an expression for the equation of motion of the buoy. The buoy will first rise, but after it gets enough water in it for the weights of the (buoy + the water inside) to exceed the buoyancy,  $[(4/3) \pi R^3 \rho g]$ , the buoy will fall.  $G$  is the acceleration of gravity. You may assume that the buoy never breaks the free surface. Also assume that the drag coefficient of the buoy (based on frontal area),  $C_d$ , is constant and the walls of the buoy are thin (so the total volume of water the buoy can hold is  $(4/3) \pi R^3$ ).

- Give an expression for the equation of motion of the buoy .
- Write a MATLAB program to integrate the above ODE by forward Euler, in order to find the velocity of the buoy as a function of time.
- Run your code for  $m_0=20 \text{ kg}$ ,  $R=0.25 \text{ m}$ ,  $Q=0.5 \text{ kg/s}$ ,  $C_d = 0.0$ ,  $\rho = 1024 \text{ kg/m}^3$ , and plot velocity versus time from  $t=0$  to  $t=350\text{s}$ , for a time step of  $\Delta t=0.2\text{s}$
- Run your code once more for  $C_d=0.5$ . Does the inclusion of viscous drag give more realistic results?
- Modify your program to use backward Euler for the same time step. Plot the results and compare to forward Euler, especially at the times where the velocity slope becomes zero.
- Decrease the time step and observe the convergence of the results. Increase the time step and observe the stability problems.
- Use a 4<sup>th</sup> order Runge-Kutta method to solve the above problem. Observe the improvement in accuracy for the same time step ( $0.2\text{s}$ ).
- Finally, replace the Runge-Kutta method by a predictor-corrector method. Use forward Euler for the prediction step and the trapezoidal rule for the correction step. What is the advantage of this method?

## Sea Waves

2. Consider a linear 2-dimensional wave that has a wavelength of 10 m. If this wave is viewed from a moving coordinate system, what is the speed of the coordinate system such that the wave field is steady?

Consider a wave field made up of four linear waves having the following characteristics:

- Wave 1: length=6m, height=0.25m, peak at  $x=1$ m
- Wave 2: length=8m, height=0.35m, peak at  $x=3$ m
- Wave 3: length=12m, height=0.5m, peak at  $x=2$ m
- Wave 4: length=17m, height=0.6m, peak at  $x=8$ m

Write a MATLAB m-script to evaluate this wave computationally, plot it very carefully, and determine the  $x$ -positions of its first three maxima at locations greater than  $x=0$ .

3. If the class lectures on May 1, 2003 or earlier describe wave spectra, do the following problem. If the lecture does not include wave spectra, it is not necessary to do this problem.

Consider a wave spectrum given by:

$$S(\omega) = \frac{0.78}{\omega^5} \exp\left[-0.74\left(\frac{1.5}{\omega}\right)^4\right]$$

where the units of  $S$  are  $\text{m}^2\text{s}$  because the 0.78 has units. The value comes from  $0.78 = A$ , where  $A=8.1 \times 10^{-3} g^2$  where  $g$  is the acceleration due to gravity.

Determine the rms sea surface elevation by analytic integration.