MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Particle Kinematics

Lagrangian and Eulerian Frames - Material Derivatives

The stream function — which will be discussed in more detail later in the course — in cylindrical co-ordinates $(r,\,\theta)$ for flow past a circular cylinder of radius a with clockwise circulation Γ is given by

$$\psi(r,\theta) = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right) \tag{1}$$

a) Write the stream function $\psi(x,y)$ in Cartesian co-ordinates, and find the components of the velocity u_x and u_y in the x and y directions.

Hint: The stream function is defined in terms of the velocity components as

$$u_x = \frac{\partial\psi(x,y)}{\partial y} \tag{2}$$

$$u_y = -\frac{\partial\psi(x,y)}{\partial x} \tag{3}$$

b) Derive the ordinary differential equations that govern the particle path lines.

c) Find the equation for the particle trajectory passing through the point $r = 2a$, $\theta = 0$ (or equivalently, $x = 2a$, $y = 0$).

d) Show that a particle on the surface of the cylinder always stays on the cylinder. Find the tangential velocity component of such a particle, and determine the stagnation points.

Hint: In cylindrical co-ordinates,

$$u_r = \frac{1}{r}\frac{\partial\psi(r,\theta)}{\partial\theta} \tag{4}$$

$$u_\theta = -\frac{\partial\psi(r,\theta)}{\partial r} \tag{5}$$

e) Sketch the stream lines for the case $\Gamma > 4\pi a U$. What happens as $r \to \infty$?

Solution:

a) We have $\psi(r,\theta) = U\left(r - \frac{a^2}{r}\right)\sin\theta + \frac{\Gamma}{2\pi}\ln\left(\frac{r}{a}\right)$

To convert to Cartesian coordinates, we make the following transformations:
$r \to \sqrt{x^2+y^2}$; $\sin\theta \to \frac{y}{\sqrt{x^2+y^2}}$; $\cos\theta \to \frac{x}{\sqrt{x^2+y^2}}$

$$\begin{aligned} \therefore \psi(x,y) &= U\left(\sqrt{x^2+y^2} - \frac{a^2}{\sqrt{x^2+y^2}}\right)\frac{y}{\sqrt{x^2+y^2}} + \frac{\Gamma}{2\pi}\ln\left(\frac{\sqrt{x^2+y^2}}{a}\right) \\ &= U\left(1 - \frac{a^2}{x^2+y^2}\right)y + \frac{\Gamma}{4\pi}\ln\left(\frac{x^2+y^2}{a^2}\right) \end{aligned} \tag{6}$$

To find $u_x(x,y)$ and $u_y(x,y)$, we use the equations 2 and 3. Upon carrying out the differentiation we have

$$u_x = U\left(1 + \frac{a^2(y^2-x^2)}{(x^2+y^2)^2}\right) + \frac{\Gamma}{2\pi}\frac{y}{x^2+y^2} \tag{7}$$

$$u_y = -\frac{2a^2xy}{(x^2+y^2)^2}U + \frac{\Gamma}{2\pi}\frac{x}{x^2+y^2} \tag{8}$$

b) A pathline is the locus of points through which a particle of a fixed identity has traveled

Let $X(t; x_0, y_0)$ and $Y(t; x_0, y_0)$ be the Lagrangian coordinate position of a fluid particle. x_0 and y_0 are parameters that refer to the initial position of the particle. We now have

$$\frac{dX}{dt} = u_x(X,Y) = u_x(X(t;x_0,y_0), Y(t;x_0,y_0))$$
$$\frac{dY}{dt} = u_y(X,Y) = u_x(X(t;x_0,y_0), Y(t;x_0,y_0))$$

We also have the initial conditions $X(t=0;x_0,y_0) = x_0$ and $Y(t=0;x_0,y_0) = y_0$. These first order linear differential equations can be solved to find the pathlines.

Another way to arrive at the above equations is to use the material derivative. We know that

$$\frac{DF}{Dt} = \frac{\partial F}{\partial t} + \mathbf{v}\cdot\nabla F \tag{9}$$

For consistency, the velocity expressed in the Lagrangian frame should equal the velocity expressed in the Eulerian frame at time t, and hence $\mathbf{v} = u_x\hat{\mathbf{e}}_x + u_y\hat{\mathbf{e}}_y$

Setting $F = x$, we get

$$\frac{Dx}{Dt} = \underbrace{\left(\frac{\partial x}{\partial t}\right)_{x,y}}_{0} + u_x\underbrace{\frac{\partial x}{\partial x}}_{1} + u_y\underbrace{\frac{\partial x}{\partial y}}_{0}$$

Therefore $\frac{Dx}{Dt} = u_x$ and similarly, $\frac{Dy}{Dt} = u_y$

c) We know that the value of the stream function must be constant along a streamline. Therefore, we first find the value of the stream function at the point $(r = 2a, \theta = 0)$ and we get

$$\psi(r, \theta)|_{(2a,0)} = \frac{\Gamma}{2\pi} \ln 2 \tag{10}$$

Thus, any particle $(r = 2a, \theta = 0)$ must have the same value of the stream function all along its trajectory. Hence, the equation of the trajectory of a particle located at $(r = 2a, \theta = 0)$ is

$$\boxed{\psi(r, \theta) = \left(r - \frac{a^2}{r}\right) \sin\theta + \frac{\Gamma}{2\pi} \ln \frac{r}{2a} = 0}$$

d) We go back to the stream function in cylindrical polar coordinates (equation 1), and using equations 4 and 5, we calculate u_r and u_θ to be

$$u_r = U\left(1 - \frac{a^2}{r^2}\right) \cos\theta \tag{11}$$

$$u_\theta = -U\left(1 + \frac{a^2}{r^2}\right) \sin\theta - \frac{\Gamma}{2\pi r} \tag{12}$$

Clearly, at $r = a$, $u_r = 0$.

A formal way to show that a particle on the cylinder always stays on the cylinder is to check that $r(t) = a$ solves the differential equation $\frac{dr}{dt} = u_r$, and satisfies the initial condition $r(t = 0) = a$. An informal way is just to check that $u_r = 0$ whenever $r = a$, and that $u_r = 0$ implies that the value of r will not change, i.e. it always stays equal to a.

e) To sketch the velocity profiles, let us look at u_θ on the surface of the cylinder (we know that $u_r(r = a) = 0$)

$$u_\theta = -2U \sin\theta - \frac{\Gamma}{2\pi a} \tag{13}$$

$$\Rightarrow \dot{\theta} = \frac{u_\theta}{a} = -\frac{2U}{a} \sin\theta - \frac{\Gamma}{2\pi a^2} \tag{14}$$

Let us plot $\dot{\theta}$ as a function of θ for the cases $\Gamma < 4\pi aU$ and $\Gamma > 4\pi aU$, which is shown in Fig. 1.

It can be seen from equations 2 and 3 that as $r \to \infty$ $(x, y \to \infty)$, the velocity field is purely horizontal with magnitude U.

Case 1: $\Gamma < 4\pi aU$

The points A and B correspond to the points at which $\dot{\theta} = 0$ on the surface of the cylinder. At these points, the sign of $\dot{\theta}$ changes, and hence the angular velocity reverses direction. To find these points, we simply find the points where $u_\theta = 0$ at $r = a$, (i. e)

$$2U \sin\theta = \frac{\Gamma}{2\pi a}$$

$$\Rightarrow \sin\theta = -\frac{\Gamma}{2\pi a}$$

Since we know that $u_r = 0$ at $r = a$, and also $u_\theta = 0$ at points A and B, these points are stagnation points.

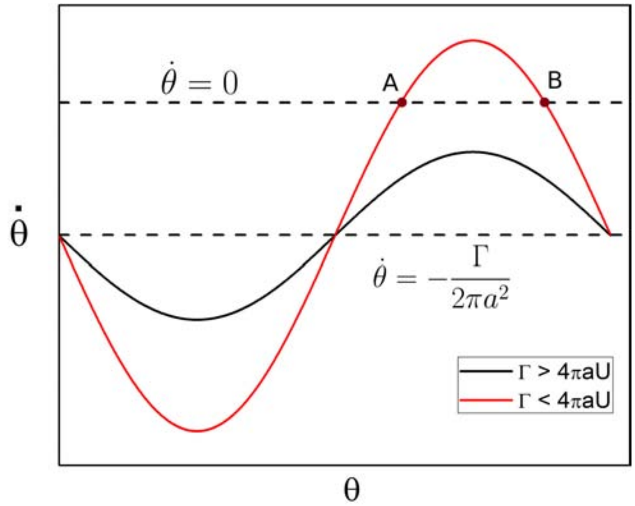

Figure 1: The variation of the angular velocity $\dot{\theta}$ at the surface of the cylinder. When $\Gamma < 4\pi aU$, A and B are stagnation points. When $\Gamma > 4\pi aU$, there are no stagnation points on the surface of the cylinder.

Case 2: $\Gamma > 4\pi aU$

To find the stagnation points, we first set $u_r = 0$

$$u_r = U\left(1 - \frac{a^2}{r^2}\right)\cos\theta = 0$$
$$\Rightarrow \cos\theta = 0 \text{ or } r = a$$

But at $r = a$, $u_\theta \neq 0$ (this can be verified by setting $r = a$ in equation 5), so there is no stagnation point on the surface of the cylinder. This means that the stagnation point lies away from the surface of the cylinder. We also know that this stagnation point must lie along the vertical line passing through the center of the cylinder because $\cos\theta = 0 \Rightarrow \theta = 90^\circ$ or $\theta = 270^\circ$ ($\theta = 0$ is the horizontal line through the center of the cylinder).

To find this stagnation point, we use $\cos\theta = 0$ ($\sin\theta = \pm 1$)and set $u_\theta = 0$ solve for r.

$$u_\theta = 0 \Rightarrow U\left(1 + \frac{a^2}{r^2}\right) = \frac{\Gamma}{2\pi r} \tag{15}$$

We have used $\sin\theta = +1$ in the above equation because $\sin\theta = -1$ has no real root. Solving this quadratic equation we find that

$$r = \frac{1}{4\pi U}\left(\Gamma + \sqrt{\Gamma^2 - (4\pi aU)^2}\right) \tag{16}$$

The other root lies inside the cylinder and is therefore discarded. A sketch of the flow profiles for the cases $\Gamma < 4\pi aU$, $\Gamma = 4\pi aU$, $\Gamma > 4\pi aU$, is given on Page 181 of the course textbook, and has been reproduced here for convenience.

□

Problem Solution by AJ, KT, Fall 2011

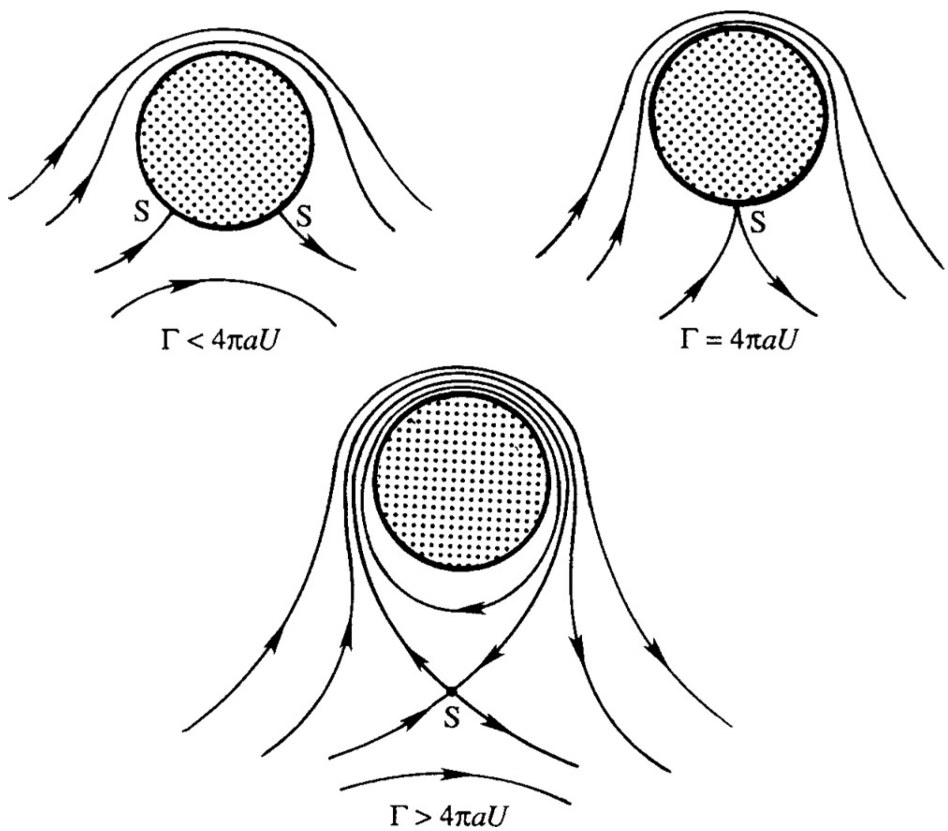

Figure 2: Different regimes of flow past a circular cylinder with circulation. Adapted from Fluid Mechanics 4th ed., P. K. Kundu and I. M. Cohen, Academic Press, 2008.

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