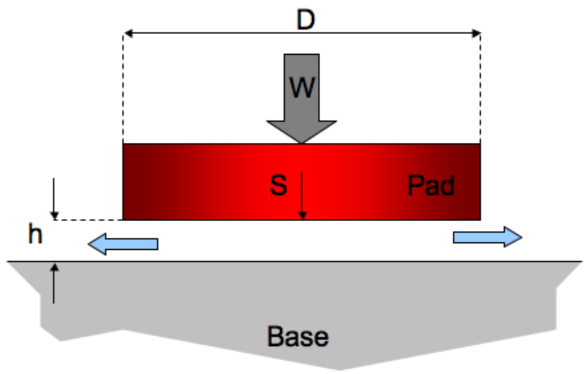
MIT Department of Mechanical Engineering
2.25 Advanced Fluid Mechanics

Problem 6.21

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin



The sketch shows a circular bearing pad which rests on a flat base through the intermediary of a film of viscous liquid of instantaneous thickness $h(t)$. The load W causes the pad to sink slowly at the speed S, and this squeezes the liquid out from under the pad. Assume that $h \ll D$, that the viscosity is very high, and that the speed S is very small.

- (a) Making approximations (state them precisely) consistent with theses assumptions, show that the settling speed is

$$S = \frac{32}{3\pi}\frac{Wh^3}{\mu D^4}. \tag{6.21a}$$

- (b) An apparatus with two very flat plates of 0.3 m diameter carries a load of 100 kg on a film 0.003 cm thick. If the liquid is a heavy oil with a kinematic viscosity of 10 $\frac{cm^2}{s}$ and a density of 0.93 $\frac{gm}{cm^3}$, estimate the speed S.

- (c) If the load W is constant, and the gap width is h_0 at time zero, show that the width h varies with time accordingly to

$$\frac{h}{h_0} = \left[1 + \frac{64}{3\pi}\frac{Wh_0^2}{\mu D^4}t\right]^{-\frac{1}{2}}. \tag{6.21b}$$

- (d) Calculate, for the initial conditions of part (b), the time (in hours) required for the gap width to be decreased to half its initial value.

- (e) Suppose now that the initial thickness is h_0, and that a constant upward force F pulls the disk away from the base. Show that the disk will be pulled away in a time

$$t_\infty = \frac{3\pi}{64}\frac{\mu D^4}{h_0^2 F}. \tag{6.21c}$$

NB When h_0 is very small, the time t_∞ is very large. This is the basis for the phenomenon of viscous adhesion, e.g., adhesives such as Scotch tape, or the apparent adhesion of accurately-ground metal surfaces.

Solution:

- (a) In terms of the flow geometry, the problem is similar to 6.3; the difference here being that the flow is "unsteady". First, let's use the continuity equation to get some information about the order of magnitude of the velocities, [1]

$$\frac{1}{r}\frac{\partial}{\partial r}(ru_r)+\frac{\partial u_z}{\partial z}=0, \Rightarrow \frac{u_r}{r}\sim\frac{u_z}{z}. \tag{6.21d}$$

Now, let's go to the NS equations. Writing the NS equations in cylindrical gap between the bearing pad and ground,

$$\rho\left(\underbrace{\frac{\partial u}{\partial t}}_{I}+\underbrace{u_r\frac{\partial u_r}{\partial r}}_{II}+\underbrace{\frac{u_\theta}{r}\frac{\partial u_r}{\partial \theta}}_{=0}+\underbrace{u_z\frac{\partial u_r}{\partial z}-\frac{u_\theta^2}{r}}_{III}\right)=-\frac{\partial p}{\partial r}+\mu\left[\underbrace{\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u_r}{\partial r}\right)}_{IV}+\underbrace{\frac{1}{r^2}\frac{\partial^2 u_r}{\partial \theta^2}}_{=0}+\underbrace{\frac{\partial^2 u_r}{\partial z^2}}_{V}-\underbrace{\frac{u_r}{r^2}}_{VI}-\underbrace{\frac{2}{r^2}\frac{\partial u_\theta}{\partial \theta}}_{=0}\right] \tag{6.21e}$$

where we relabeled the terms to further make an order of magnitude estimation, also notice that due to symmetry, the terms $\frac{\partial}{\partial\theta}$. Let's start comparing the terms IV and V, [2]

$$\frac{IV}{V}\sim\frac{h^2}{D^2}\ll 1, \tag{6.21f}$$

so we can neglect the term IV compared with term V. Also, notice that $VI\sim IV$, then VI can be neglected when compared to V. Now, let's compare II to V, then

$$\frac{II}{V}\sim\frac{\rho v_r D}{\mu}\frac{h^2}{D^2}\ll 1, \tag{6.21g}$$

then II can also be neglected when compared to V. Now, when comparing III to V, we first notice that $II\sim III$ since

$$\frac{II}{III}\sim\frac{u_r/r}{u_z/z}\sim\frac{u_z/z}{u_z/z}\sim 1, \tag{6.21h}$$

where we have used the information obtained from mass conservation. Hence, III vanishes when compared to V.

Now, when comparing I to V, we have

$$\frac{I}{V}\sim\frac{\rho h^2}{\mu\tau}, \tag{6.21i}$$

where τ is the time scale involved in the process. The source of unsteadyness is the pad settling down, which renders u_r and other flow variables time-dependent. Hence,

$$\tau\sim\frac{h}{S}\Rightarrow\sim\frac{\rho S h}{\mu}, \tag{6.21j}$$

Since it is given that S is very small and also h is small, we can safely assume that I can be neglected compared to V. Hence we have the NS equation as

[1] By now, you should be familiar with this method of obtaining extra information that can be quite useful when comparing terms in the NS equations.

[2] We'll compare the terms with term V because, since the gap is small, this is likely to be the largest derivative.

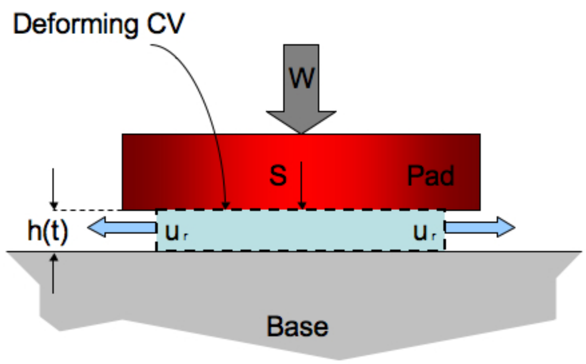
$$-\frac{\partial p}{\partial r}+\mu\frac{\partial^2 u_r}{\partial z^2}=0. \tag{6.21k}$$

Integrating we get $u_r(z)$,

$$u_r=\frac{1}{2\mu}\Big(-\frac{\partial p}{\partial r}\Big)(zh-z^2)\Rightarrow \tag{6.21l}$$

$$Q(r)=2\pi r\int_0^h u_r(z)dz=\frac{h^3\pi r}{6\mu}\Big(-\frac{\partial p}{\partial r}\Big)=\frac{h^3\pi r}{6\mu}\Big(-\frac{dp}{dr}\Big) \tag{6.21m}$$

Note that Q is a function of r, since the settling down of the pad drives greater and greater flow rates as r increases! This can be verified by applying mass conservation in a cylindrical CV, as shown. The height of this CV changes as $h(t)$.



$$\frac{\partial}{\partial t}\int_{CV}\rho dV+\int_{CV}\rho(\underline{V}-\underline{V}_{cs})\cdot\underline{n}dA=0, \tag{6.21n}$$

$$\Rightarrow\frac{\partial}{\partial t}(\pi r^2 h)+2\pi rhu_r=0,\Rightarrow -\pi r^2\frac{dh}{dt}=2\pi rhu_r=Q(r). \tag{6.21o}$$

Since $\frac{dh}{dt}=-S$, we have

$$Q(r)=\pi r^2 S. \tag{6.21p}$$

Now, we can find the value of the pressure gradient using the flow, then

$$-\frac{dp}{dr}=\frac{6\mu Q}{h^3\pi r}=\frac{6\mu\pi r^2 S}{h^3\pi r}=\frac{6\mu rS}{h},\Rightarrow \tag{6.21q}$$

$$dp=-\frac{6\mu Sr}{h^3}dr,\Rightarrow p(r)=\frac{3\mu S}{h^3}\Big(\frac{D^2}{4}-r^2\Big), \tag{6.21r}$$

where the BC used is $p(D/2)=0$ (gauge pressure).

Now, we can perform a vertical force balance on the pad,

$$W=\frac{3\mu S}{h^3}\int_0^{\frac{D}{2}}\Big(\frac{D^2}{4}-r^2\Big)2\pi rdr=\frac{3\pi\mu SD^4}{32h^3}, \tag{6.21s}$$

then, we can finally get the velocity

$$S = \frac{32Wh^3}{3\pi\mu D^4}, \tag{6.21t}$$

- (b) Plugging in the numbers, we obtain

$$S = \frac{32 \times 100 \times \times 27 \times 10^{-15}}{3\pi \times 0.93 \times 81 \times 10^{-4}} = 1.2 \times 10^{-9}[m/s] \tag{6.21u}$$

very small, which agrees with our assumptions.

- (c) From the velocity equation, we can integrate to obtain the displacement

$$S = -\frac{dh}{dt} = \frac{32Wh^3}{3\pi\mu D^4} \Rightarrow -\int_{h_0}^{h} \frac{dh'}{h'^3} = \int_0^t \frac{32W}{3\pi\mu D^4} dt', \tag{6.21v}$$

then, upon integration we have

$$-\left[-\frac{1}{2}\frac{1}{h'^2}\right]_{h_0}^{h} = \frac{32Wt'}{3\pi\mu D^4}\Bigg|_0^t \Rightarrow \frac{1}{h^2} - \frac{1}{h_0^2} = \frac{64Wt}{3\pi\mu D^4} \Rightarrow \frac{1}{h^2} = \frac{1}{h_0^2}\left[1 + \frac{64Wh_0^2}{3\pi\mu D^4}t\right] \tag{6.21w}$$

which gives

$$\Rightarrow \frac{h}{h_0} = \left[1 + \frac{64Wh_0^2}{3\pi\mu D^4}t\right]^{-\frac{1}{2}}. \tag{6.21x}$$

- (d) Using $2h = h_0$, and plugging in the variables, we get the required time as 10.4 hours.
- (e) Since $S' = \frac{dh}{dt}$, instead of $-\frac{dh}{dt}$, we have

$$\frac{1}{h_0^2} - \frac{1}{h^2} = \frac{64Ft}{3\pi\mu D^4}. \tag{6.21y}$$

As the disk is pulled away, $h \to \infty$ then for this limit,

$$t_\infty = \frac{3\pi\mu D^4}{64h_0^2 F}. \tag{6.21z}$$

□

Problem Solution by MK/MC(Updated), Fall 2008

