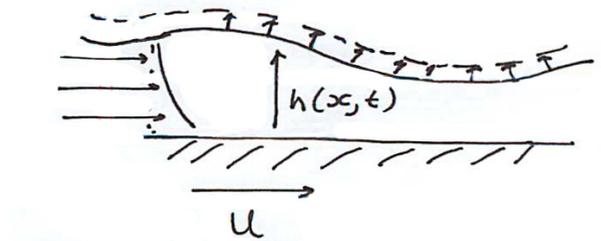**THE GENERAL FORM OF REYNOLDS EQUATIONS**

- For a general (moving) boundary with $h(x,t)$
- The flow can be unsteady but is fully-developed locally.

1. Beginning from the locally fully-developed flow derived in class, for which we required that $\left(\dfrac{h}{L}\right)^2 \ll 1$, $\mathrm{Re}_L\left(\dfrac{h}{L}\right)^2 \ll 1$, we know that the velocity profile in the gap is:

$$v_x(x,y,t) = U\left(1 - \frac{y}{h(x,t)}\right) + \frac{h(x,t)^2}{2\mu}\left(-\frac{dp}{dx}\right)\left[\frac{y}{h(x,t)} - \left(\frac{y}{h(x,t)}\right)^2\right] \tag{1}$$

$$\underbrace{\qquad}_{\substack{\text{Couette} \\ \text{(wall driven)}}} \quad \underbrace{\qquad}_{\substack{\text{Poiseuille} \\ \text{(pressure driven)}}}$$

Integrating this expression gives an expression for the local flow rate q' at any given slice:

$$q'(x,t) \equiv \int_0^{h(x,t)} v_x\, dy = \frac{h(x,t)^3}{12\mu}\left(-\frac{\partial p}{\partial x}\right) + \frac{1}{2}Uh(x,t) \tag{2}$$

**Note that this may not be constant along the channel because of the squeezing flow induced by vertical motion of the channel boundary at $h(x,t)$.

2. We combine with this with an integrated analysis of the flow in the vertical direction, and the use of what is commonly called the "kinematic boundary condition" at the upper plate.

– This links the displacement in the boundary motion to the local fluid flow rate
– Essentially we apply conservation of mass to the thin strip of fluid $h(x,t)dx$

a). We start with $\dfrac{\partial v_x}{\partial x} + \dfrac{\partial v_y}{\partial y} = 0$, integrate in y to obtain

$$\int_0^{h(x)} \frac{\partial v_x}{\partial x}\, dx + \int_0^{h(x)} \frac{\partial v_y}{\partial y}\, dy = 0 \quad (3a)$$

$$(?) + V_y\Big|_{y=0}^{y=h(x,t)} = 0 \qquad (3b)$$
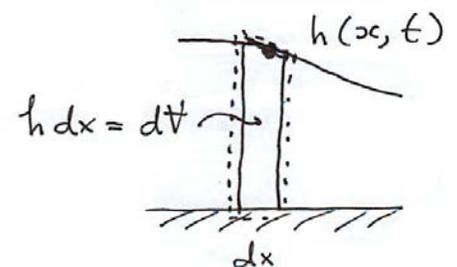
The second integral introduces the boundary conditions for the vertical rate of displacement of the upper and lower surfaces (which may be stationary or moving BCs).

To find the first term, labeled $(?)$ in eq. 3, we need to use the Leibnitz Theorem, in the form:

$$\int_a^b \frac{\partial f}{\partial x}\, dy = \frac{\partial}{\partial x}\int_a^b f\, dy - \left\{f\Big|_{y=b}\frac{db}{dx} - f\Big|_{y=a}\frac{da}{dx}\right\}$$

recognizing that here $f = v_x$, $a = 0$ and $b = h(x)$, we can write that

$$\int_0^{h(x)} \frac{\partial v_x}{\partial x} dy = \frac{d}{dx} \int_0^h v_x \, dy - \left\{ \frac{dh}{dx} v_x \Big|_{y=h} - \frac{d(0)}{dx} U \Big|_{y=0} \right\} \qquad (4)$$

The first term on the right-hand side is nothing but the variation in the flow rate of fluid along the gap (which again we re-emphasize might not be constant because of the squeezing induced by the top plate etc).

• The terms in { } again have to be evaluated based on the actual boundary conditions of the specific problem at the top and bottom plates.

Combining eq.(3) with 2(b) results in the expression: $\dfrac{dq'}{dx} - \dfrac{dh}{dx} U \Big|_{y=h} + V \Big|_{y=h} = 0$ (5)

[Alternately you can apply a control volume analysis directly to a thin slice of the form shown in the sketch above to arrive at exactly the same final result].

b) To simplify this result, we can recognize that the equation for the location of the free surface is simply the solution of the equation: $y - h(x,t) = 0$. Since *material points on the surface stay on the surface* as it moves and deforms, we can take the convected derivative of this expression so that

$$\frac{D}{Dt}(y - h(x,t)) = 0 \qquad \Rightarrow \qquad -\frac{\partial h}{\partial t} - v_x \Big|_{y=h} \frac{dh}{dx} + v_y \Big|_{y=h} = 0 \qquad (6)$$

This is commonly known as *the kinematic boundary condition* and it is of exactly the same form as the expression derived in eq. (5). Combining eq. (5) and (6) thus results in the following *much simpler* expression:

$$\frac{dq'}{dx} + \frac{\partial h}{\partial t} = 0 \qquad (7)$$

3) We can now combine this with the result obtained in eq. (2) to obtain the following nonlinear coupled PDE:

$$\frac{1}{\mu} \frac{\partial}{\partial x} \left(h^3 \frac{\partial p}{\partial x} \right) = 6U \frac{\partial h}{\partial x} + 12 \frac{\partial h}{\partial t} \qquad (8)$$

This is referred to as the general form of <u>Reynolds Lubrication Equation.</u> If the shape of the boundary and it's displacement are specified (i.e. *h(x,t)* is given) then it gives a second order differential equation that can be solved for the pressure field *p(x,t)* (subject to appropriate boundary conditions on the pressure field at the edge of the fluid film). Alternatively it can be combined with a global force balance on a moving block or plate or other object that has a thin film of viscous fluid underneath it to develop a differential equation for the motion of the block. It is thus the starting point for many different research topics as well as homework or qualifying problems!

There are many, many generalizations, e.g.
- cylindrical coordinates
- compressible fluids
- 'suction' or 'porous wall' boundary conditions
- stretching surfaces, collapsing, bending surfaces
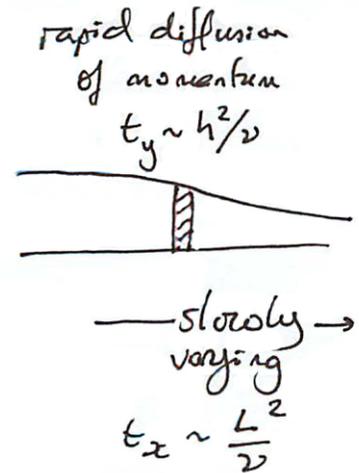
*All collapse down to the same basic physics, which is the key thing to internalize:

(i) nondimensionalize governing equations of motion.

(ii) drop "small" terms i.e. $0\left[\operatorname{Re}_L\left(\dfrac{H}{L}\right)^2\right]...$

(iii) solve locally fully-developed viscous flow in slice dx

(iv) integrate across the slice (i.e. the "fast" direction) to find an integrated quantity such as the local flow rate per unit depth, $q'(x)$

(v) use conservation of mass or other appropriate BCs to integrate along the "slow" direction, along which integrated quantity $q'(x)$ varies

Remember that effectively we have the following <u>two independent constraints</u>:

$$t_y \ll \frac{L}{V} \qquad and \qquad t_y \ll t_x$$

i.e. the time-scale for diffusion of vorticity information *across the thin gap* is always much shorter than the time required for the information to be convected along the gap by the fluid flow (with convection time $t_{conv} \sim L/V$) or to diffuse along the gap (with a characteristic time scale $t_x \sim L^2/v$). This is why we say the flow is *locally fully-developed.*

- The second constraint is the same as saying: $h^2/L^2 \ll 1$
- The first constraint can be rearranged several different ways:

$$\rho\frac{h^2}{\mu} \ll \frac{L}{V} \quad\Rightarrow\quad \frac{\rho V h^2}{\mu L} \ll 1 \quad or \quad \left(\frac{\rho V h}{\mu}\right)\left(\frac{h}{L}\right) \ll 1 \quad or \quad \left(\frac{\rho V L}{\mu}\right)\left(\frac{h}{L}\right)^2 \ll 1$$

rapid diffusion
of momentum
$t_y \sim h^2/v$

—slowly →
varying

$t_x \sim \dfrac{L^2}{v}$

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