Equation of Motion in Streamline Coordinates

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2.25 Advanced Fluid Mechanics

Euler's equation expresses the relationship between the velocity and the pressure fields in inviscid flow. Written in terms of streamline coordinates, this equation gives information about not only about the pressure-velocity relationship along a streamline (Bernoulli's equation), but also about how these quantities are related as one moves in the direction transverse to the streamlines. The transverse relationship is often overlooked in textbooks, but is every bit as important for understanding many important flow phenomena, a good example being how lift is generated on wings.

A streamline is a line drawn *at a given instant in time* so that its tangent is at every point in the direction of the local fluid velocity (Fig. 1). Streamlines indicate local flow direction, not speed, which usually varies along a streamline. In steady flow the streamline pattern remains fixed with time; in unsteady flow the streamline pattern may change from instant to instant.

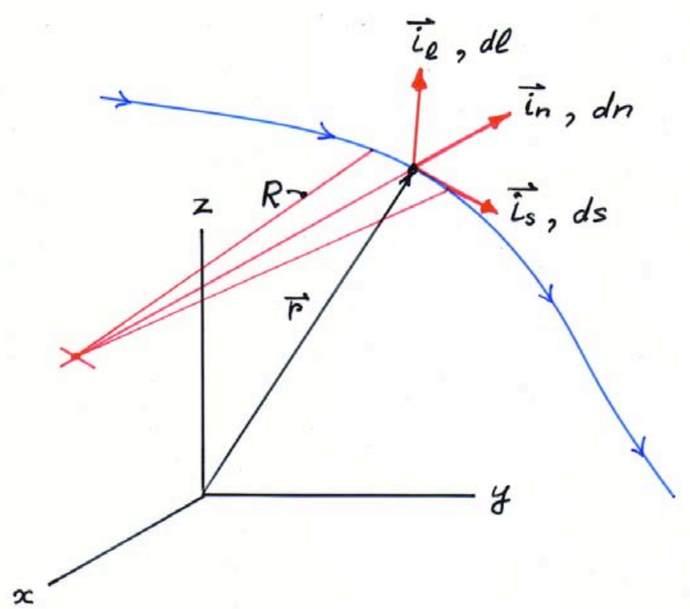

Fig. 1: Streamline coordinates

In what follows, we simplify the exposition by considering only *steady, inviscid* flows with a *conservative body forces* (of which gravity is an example). A conservative force per unit mass $\vec{G}$ is one that may be expressed as the gradient of a time-invariant scalar function,

$$\vec{G} = -\nabla U(\vec{r}) \, , \tag{1}$$

and the steady-state Euler equation reduces to

$$\vec{V} \cdot \nabla \vec{V} = -\frac{1}{\rho} \nabla p - \nabla U(\bar{r}) \,. \tag{2}$$

A uniform gravitational force per unit mass *g* pointing in the negative z direction is represented by the potential

$$U = gz \,. \tag{3}$$

A streamline coordinate system is not chosen arbitrarily, but follows from the velocity field (which, we note, is not known à priori). Associated uniquely with any point $\bar{r}$ and time *t* in a flow field are (Fig. 2): the streamline that passes through the point (streamlines cannot cross), the streamline's local radius of curvature *R* and center of curvature, and the following triad of orthogonal unit vectors:

$\vec{i}_s$: in the flow direction
$\vec{i}_n$: in the normal direction, *away* from the local center of curvature
$\vec{i}_l$: in the bi-normal direction, ($\vec{i}_l = \vec{i}_s \times \vec{i}_n$).

The unit vectors define incremental distance *ds* measured along the streamline in the flow direction, *dn* measured in the normal direction, away from the center of curvature, and d*l* measured in the bi-normal direction. The radius of curvature *R* is defined as positive if $\vec{i}_n$ points away from the center of curvature, and negative if $\vec{i}_n$ points toward it. The unit vectors, the radius of curvature, and the center of curvature all change from point to point and in unsteady flows from time to time, depending on the velocity field.

To transform Euler's equation into streamline coordinates, we note that in those coordinates[1],

$$\nabla = \vec{i}_s \frac{\partial}{\partial s} + \vec{i}_n \frac{\partial}{\partial n} + \vec{i} \frac{\partial}{\partial l} \tag{4}$$

and

$$\vec{V} = \vec{i}_s V \tag{5}$$

where *V* is the magnitude of the velocity vector $\vec{V}$. From (4) and (5),

$$\vec{V} \cdot \nabla = V \frac{\partial}{\partial s} \tag{6}$$

and thus

[1] The gradient of a scalar function $f(s,n,l)$ is defined by $\nabla f \cdot d\bar{r} \equiv df(s,n,l) = \frac{\partial f}{\partial s} ds + \frac{\partial f}{\partial n} dn + \frac{\partial f}{\partial l} dl$. Equation (4) follows from this definition and the expression $d\bar{r} = \bar{i}_s ds + \bar{i}_n dn + \bar{i}_l dl$ for an incremental displacement in streamline coordinates.

$$(\vec{V}\cdot\nabla)\vec{V} = V\frac{\partial}{\partial s}\left(V\vec{i}_s\right) = \vec{i}_s\frac{\partial}{\partial s}\left(\frac{V^2}{2}\right) + V^2\frac{\partial \vec{i}_s}{\partial s} \ . \tag{7}$$

The unit vector in the last term of (7) changes orientation as one moves along the streamline. The change $d\vec{i}_s$ in $\vec{i}_s$ from s to $s+ds$ is obtained with the construction shown in Fig. 2 as

$$d\vec{i}_s = -\vec{i}_n d\theta = -\vec{i}_n\frac{ds}{R} \tag{8}$$

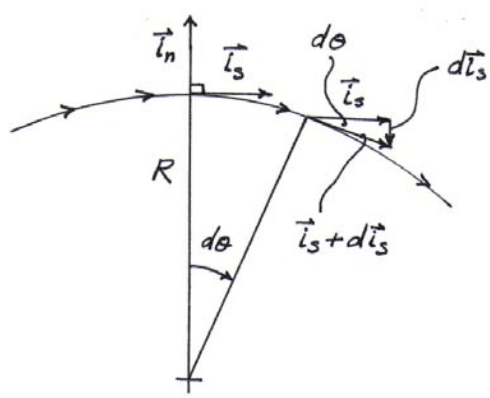

Fig. 2: Incremental change in the streamwise unit vector from s to $s+ds$.

from which we see that

$$\frac{\partial \vec{i}_s}{\partial s} = -\frac{\vec{i}_n}{R} \tag{9}$$

Using (9) in (7), we obtain the convective acceleration as

$$(\vec{V}\cdot\nabla)\vec{V} = \vec{i}_s\frac{\partial}{\partial s}\left(\frac{V^2}{2}\right) - \vec{i}_n\frac{V^2}{R} \tag{10}$$

The first term on the right is the convective acceleration in the direction of the velocity, and the second is the centripetal acceleration, toward the center of curvature.

The pressure gradient in streamline coordinates is

$$\nabla p = \vec{i}_s\frac{\partial p}{\partial s} + \vec{i}_n\frac{\partial p}{\partial n} + \vec{i}_l\frac{\partial p}{\partial l} \tag{11}$$

Using (10) and (11) in (2), we obtain the equation of motion in streamline coordinates for *steady*, *inviscid* flow as

$$s\text{-direction:}\qquad \frac{\partial}{\partial s}\left(\frac{V^2}{2}\right) = -\frac{1}{\rho}\frac{\partial p}{\partial s} - \frac{\partial U}{\partial s} \tag{12}$$

$$n\text{-direction:}\qquad -\frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} - \frac{\partial U}{\partial n} \tag{13}$$

$$l\text{-direction:}\qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial l} - \frac{\partial U}{\partial l} \tag{14}$$

In a *uniform gravitational field U=gz* and these equations read

$$s\text{-direction:}\qquad \frac{\partial}{\partial s}\left(\tfrac{1}{2}V^2\right) = -\frac{1}{\rho}\frac{\partial p}{\partial s} - g\frac{\partial z}{\partial s} \tag{15}$$

$$n\text{-direction:}\qquad -\frac{V^2}{R} = -\frac{1}{\rho}\frac{\partial p}{\partial n} - g\frac{\partial z}{\partial n} \tag{16}$$

$$l\text{-direction:}\qquad 0 = -\frac{1}{\rho}\frac{\partial p}{\partial l} - g\frac{\partial z}{\partial l} \tag{17}$$

For c*onstant-density flow* in a *uniform gravitational field*, the equations simplify further to

$$s\text{-direction:}\qquad \frac{\partial}{\partial s}\left(p + \rho g z + \frac{\rho V^2}{2}\right) = 0 \tag{18}$$

$$n\text{-direction:}\qquad \frac{\partial}{\partial n}\left(p + \rho g z\right) = \frac{\rho V^2}{R} \tag{19}$$

$$l\text{-direction:}\qquad \frac{\partial}{\partial l}\left(p + \rho g z\right) = 0 \tag{20}$$

The *s*-direction equation (18) states Bernoulli's theorem: the total pressure—the sum $p + \rho g z + \rho V^2/2$ of the static, gravitational, and dynamic pressures—remains invariant along a streamline.

The *n*-direction equation (19) states that when there is flow and the streamlines curve, the sum $p + \rho g z$ (which is constant in when the fluid is static) *increases in the n-direction*, that is, as one moves *away* from the local center of curvature.

The l-direction equation (20) states that $p + \rho g z$ remains constant for *small* steps in the binormal direction, that is, the pressure distribution is quasi-hydrostatic distribution in the l-direction.

EXAMPLE

Consider the simple case of 2D, inviscid air flow over a smooth hill (Fig. 3). Far upstream of the hill the incident velocity is uniform at V_∞. The hill deflects the air around it, and a uniform flow is again established far downstream. Far upstream, above, and downstream of the hill, the pressure is constant at p_∞ and the streamlines are straight (the hill does not perturb the flow at "infinity"). We shall assume that gravitational effects are negligible (the medium is air and the hill's elevation is modest) and the free stream's Mach number is small, so that and the density can be taken as constant. Based on the available equations, what can we say about the pressure and velocity distributions over the hill—where is the velocity higher than V_∞, for example, and where lower?

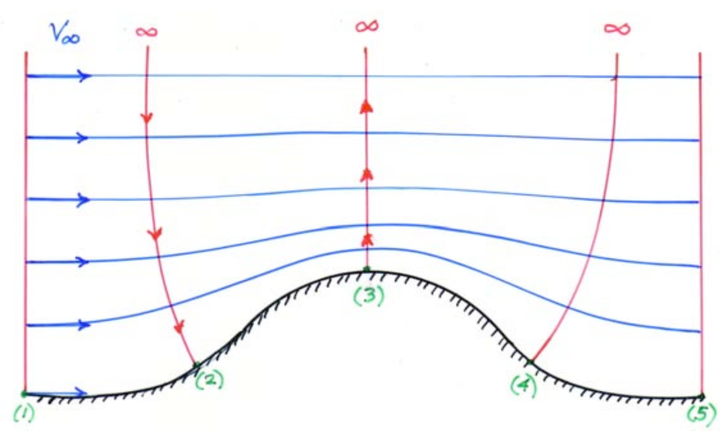

Fig. 3: Sketch of streamlines in a 2D flow over a hill.

To answer this question accurately we need to know the shapes of the streamlines throughout the flow field—or, at least, in the region that is perturbed by the hill. We don't have this information, so we proceed by drawing a rough estimate of the streamline pattern, as shown in Fig. 3. The difference between the pressure at infinity and at the top of the hill, point (3), can be estimated by integrating equation (19) along the vertical path from (3) to (∞). Since this path follows the local n-direction, $R>0$ everywhere along it. Neglecting the gravitational term, (19) gives

$$\frac{\partial p}{\partial n} = \frac{\rho V^2}{R} \tag{21}$$

from which we see that

$$p_\infty - p_3 = \int_3^\infty \frac{\rho V^2 dn}{R} > 0 \qquad (22)$$

Thus $p_3 < p_\infty$, and according to Bernoulli's equation (18), it follows that $V_3 > V_\infty$. Using similar arguments, we conclude that $p_1 = p_\infty$ and $V_1 = V_\infty$, and $p_2 > p_\infty$ and $V_2 < V_\infty$, etc.

In principle, if *R(n)* and *V(n)* can be established or estimated, the integral in (21) can be evaluated. For example if we find that the flow perturbation caused by the hill is negligible at elevations greater than some multiple of the height of the hilltop, we might write for the path from (3) to (∞)

$$R \approx R_{hill} e^{\frac{n}{H}} \qquad (23)$$

where R_{hill} is the streamlines' radius of curvature in the vicinity of the hilltop, n is measured from the top of the hill upward, and $H = \beta h$ is some multiple β of the actual height h of the hilltop, the coefficient β being an empirical number. From Bernoulli's equation (18) we also have that

$$p + \frac{\rho V^2}{2} = p_\infty + \frac{\rho V_\infty^2}{2} \quad , \qquad (24)$$

Substituting for R and V into (21) from (23) and (24), respectively, we integrate (23) and obtain

$$p_\infty - p_3 = \frac{\rho V_\infty^2}{2}\left(e^{\frac{2H}{R_{hill}}} - 1\right) \qquad (25)$$

For a low hill such that $2H << R_{hill}$, the exponential term can be expanded and (25) simplified to

$$p_\infty - p_3 \approx \frac{\rho V_\infty^2 H}{R_{hill}} \qquad (26)$$

The velocity at point (3) now follows from (24) and (25) as

$$V_3 = V_\infty e^{\frac{H}{R_{hill}}} \qquad (27)$$

or, in the same low-hill approximation as (26),

$$V_3 \approx V_\infty \left(1 + \frac{H}{R_{hill}}\right) \qquad (28)$$