Symmetry of Stress Tensor

Imagine an arbitrary fluid element which in 2-D is a rectangle with width δx_1 in the $x, 1$ direction and height δx_2 in the $y, 2$ direction; the element also has a width δx_3 in the $z, 3$ direction (Figure 1). Stresses acting on each face can be calculated using the values at point O (center of the element) and applying a Taylor series expansion in each direction:

Shear stress acting on the right wall: $\tau_{12} \mid_O + \delta\tau_a$
Shear stress acting on the left wall: $\tau_{12} \mid_O - \delta\tau_a$
Shear stress acting on the bottom wall: $\tau_{21} \mid_O - \delta\tau_b$
Shear stress acting on the top wall: $\tau_{21} \mid_O + \delta\tau_b$

in which:

$$\delta\tau_a = \frac{\partial\tau_{12}}{\partial x_1}(\delta x_1/2)$$
$$\delta\tau_b = \frac{\partial\tau_{21}}{\partial x_2}(\delta x_2/2)$$

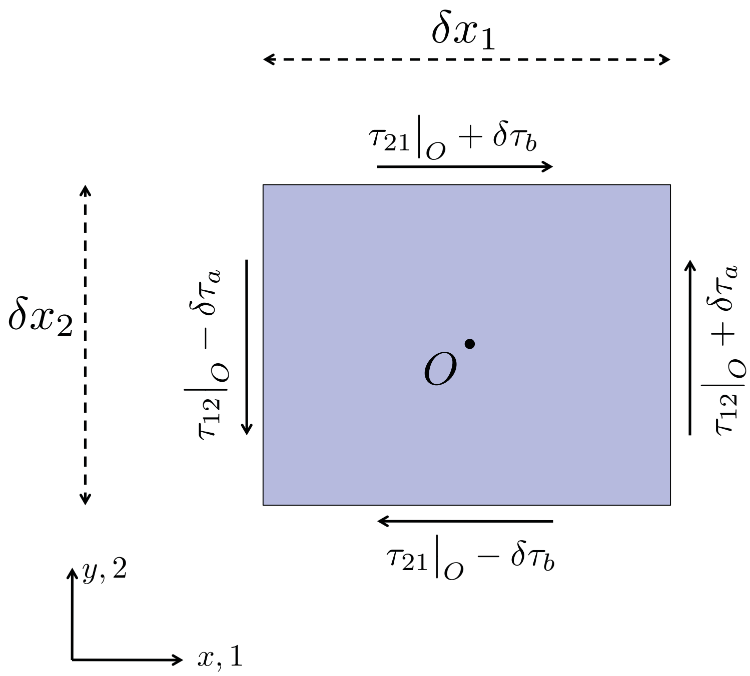

Figure 1: Taylor series expansion for the shear stresses acting on a material element of size δx_1, δx_2.

Knowing that the normal stresses acting on each plane will not lead to any net torque around axis x_3 passing through point O one can calculate the net exerted torque on the element (M_O) by accounting for all the shear stresses acting on the element:

$$\begin{aligned}\sum M_O =& (\tau_{12} + \delta\tau_a)(\delta x_2 \delta x_3)(\delta x_1/2) + (\tau_{12} - \delta\tau_a)(\delta x_2 \delta x_3)(\delta x_1/2) \\ & - (\tau_{21} + \delta\tau_b)(\delta x_1 \delta x_3)(\delta x_2/2) - (\tau_{21} - \delta\tau_b)(\delta x_1 \delta x_3)(\delta x_2/2)\end{aligned}$$

Which can be simplified to give:

$$\sum M_O = (\tau_{12} - \tau_{21})\delta x_1 \delta x_2 \delta x_3 \tag{1}$$

On the other hand we know that the following holds:

$$\sum M_O = I\dot{\omega_3}$$

in which I is the moment of inertia around x_3 axis passing through point O and for a cuboidal element it is:

$$I = \frac{\rho}{12}\delta x_1 \delta x_2 \delta x_3 (\delta x_1^2 + \delta x_2^2) \tag{2}$$

Combining (1) and (2) will result in:

$$\dot{\omega_3} = \frac{12}{\rho}\frac{\tau_{12} - \tau_{21}}{\delta x_1^2 + \delta x_2^2}$$

It is easy to see that if one shrinks the element to a very small volume (i.e. δx_1 and $\delta x_2 \to 0$) the rotational acceleration of the element ($\dot{\omega_3}$) will diverge to infinity unless the shear stress difference also tends to zero at least as fast as $\delta x_i^2 \to 0$ (thus $\tau_{12} - \tau_{21} = 0$). Since infinite rotational acceleration is not physically possible the stress tensor should be symmetric, $\tau_{ij} = \tau_{ji}$.[1]

[1] The mentioned proof is true in the absence of magneto-hydrodynamic forces or other non-conservative body forces.