MIT Department of Mechanical Engineering

2.25 Advanced Fluid Mechanics

Problem 1.10

This problem is from "Advanced Fluid Mechanics Problems" by A.H. Shapiro and A.A. Sonin

The Swiss scientist Auguste Picard developed a navigable diving vessel, the "bathyscape", to investigate the ocean at great depths (http://en.wikipedia.org/wiki/Bathyscaphe). In 1960, his son Jacques, accompanied by Lt. Don Walsh of the U.S. Navy, reached a depth of 10,916 m in the Pacific's Mariana Trench.

Suppose that the ocean is at constant temperature, has a density of 1030 kg/m^3 at sea level, and is characterized by a constant isothermal bulk compressibility

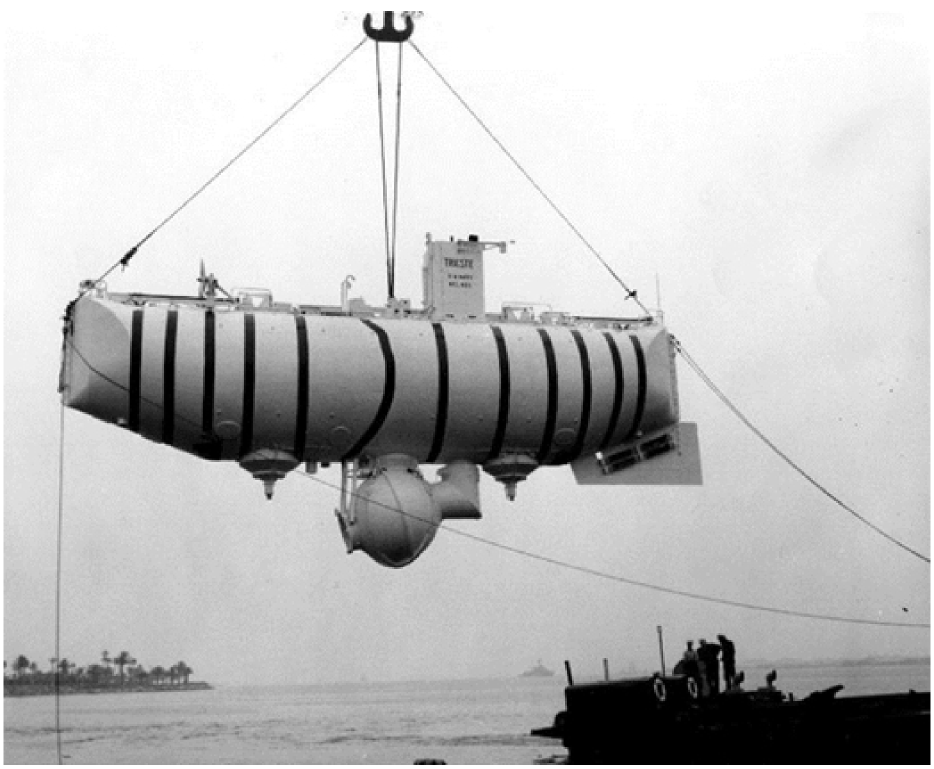
$$\kappa_T \equiv \frac{1}{\rho}\left(\frac{\partial \rho}{\partial p}\right)_T = 4.6 \times 10^{-10}\,\mathrm{m^2/N}. \tag{1.10a}$$

Compute the pressure at a depth of 11 km,

(a) assuming the density is constant at the sea level value, and

(b) taking the water's compressibility into account.

For part (b), derive an expression for the pressure as a function of depth below the surface, considering the sea level density ρ_0 and pressure p_0, as well as κ_T, as given quantities.



Solution:

(a) Assuming the density is constant at the sea level value and the pressure at sea level is $p_0 = 1.01 \times 10^5$ Pa, we find that

$$\begin{aligned} p &= p_0 - \rho g h \qquad (1.10b) \\ &= 1.01 \times 10^5\,\text{Pa} - (1030\,\text{kg/m}^3)(9.8\,\text{m/s}^2)(-11,000\,\text{m}) \\ &= 1.11 \times 10^8\,\text{Pa} \\ &= 111\,\text{MPa} \end{aligned}$$

(b) Taking the water's compressibility into account, the density of water ρ will vary with pressure p. First we will solve the compressibility equation [Eq. (1.10a)] by separating variables to get ρ in terms of p.

$$\int_{p_0}^{p} \kappa_T dp = \int_{\rho_0}^{\rho} \frac{1}{\rho} d\rho$$

$$\kappa_T (p - p_0) = \ln \rho - \ln \rho_0 = \ln \frac{\rho}{\rho_0}$$

Solving for ρ, we find:

$$\rho = \rho_0 e^{\kappa_T (p - p_0)} \qquad (1.10c)$$

At this point, you may be inclined to substitute this expression for ρ into the pressure equation [Eq. (1.10b)] used in part (a). However, we note that this pressure distribution assumes a constant density ρ (see Kundu & Cohen [K&C] pp.11). Instead, we use the more general form of the pressure gradient [Eq. (1.8) in K&C] and substitute Eq. (1.10c) to give

$$\begin{aligned} \frac{dp}{dz} &= -\rho g \\ &= -\rho_0 e^{\kappa_T (p - p_0)} g. \end{aligned}$$

Again, we separate variables and integrate:

$$\int_{p_0}^{p} e^{-\kappa_T (p - p_0)} dp = -\int_{0}^{h} \rho_0 g dz$$

$$-\frac{1}{\kappa_T} \left(e^{-\kappa_T (p - p_0)} - 1 \right) = -\rho_0 g h$$

After some algebra, we finally have

$$\begin{aligned} p &= p_0 - \frac{1}{\kappa_T} \ln(1 + \kappa_T \rho_0 g h) \qquad (1.10d) \\ &= 1.01 \times 10^5\,\text{Pa} - \frac{1}{4.6 \times 10^{-10}\,\text{m}^2/\text{N}} \ln\left[1 + (4.6 \times 10^{-10})(1030)(9.8)(-11,000)\right] \\ &= 114\,\text{MPa} \qquad (1.10e) \end{aligned}$$

As a check on our pressure equation [Eq. (1.10d)], take the limit as $x = \kappa_T \rho_0 g h$ is small. Note that $\ln(1 + x) \approx x$ for small values of x. Thus,

$$\begin{aligned} p &= p_0 - \frac{1}{\kappa_T} \ln(1 + \kappa_T \rho_0 g h) \\ &\approx p_0 - \frac{1}{\kappa_T} \kappa_T \rho_0 g h \\ &= p_0 - \rho_0 g h \end{aligned}$$

Note, the equation above is the the same as the incompressible pressure equation [Eq. (1.10b)] in part (a).

□

Problem Solution by Tony Yu (MC updated), Fall 2006

