Problem 2: Lubricated Pipelining

A common engineering challenge faced in pumping viscous crude oil over long distances is the large power consumption required to convey the oil through the pipeline. One proposed solution is to lubricate the pipeline as shown below using a thin layer of an immiscible fluid (such as water) with a lower viscosity to surround the oil and lubricate the motion. We shall model the flow as flow in a cylindrical pipe of radius R with a core of thickness R_1 consisting of very viscous liquid oil with viscosity μ_1 surrounded by a shell of water (or other low viscosity fluid) of thickness $\delta = R - R_1$ that is density matched (so that $\rho_1 = \rho_2 = \rho$) with viscosity $\mu_2 < \mu_1$. The interfacial tension between the two liquids is denoted σ. The average velocity of the oil through the pipe is denoted $\overline{v}_o = Q_{oil} / \pi R_1^2$

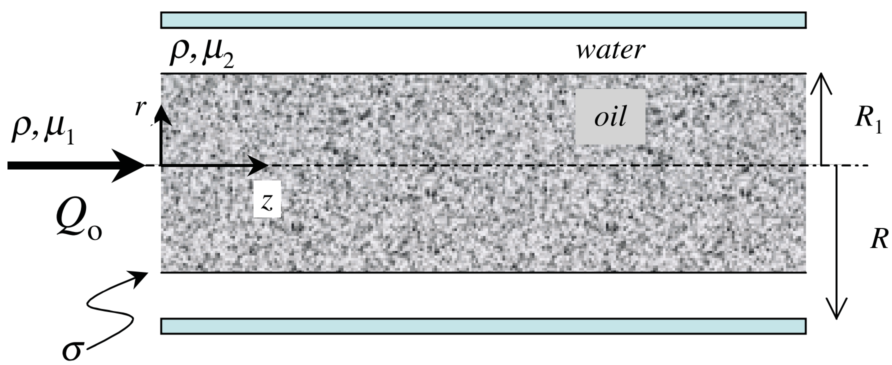
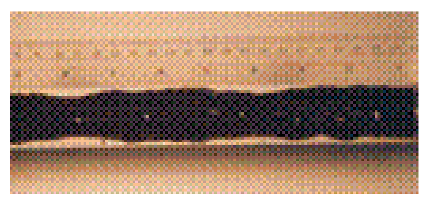


Figure 1: geometry of a lubricated pipeline

a) Although the oil-water interface shown in the figure above is depicted as planar, in reality under certain operating conditions interfacial waves may form as shown in the picture opposite:



http://www.aem.umn.edu/research/pipeline/horizontalindex.html

Use dimensional analysis to determine an appropriate dimensionless form for expressing the fully-developed pressure drop per unit length in the pipe $-\dfrac{\partial P}{\partial z} = \dfrac{\Delta P}{L}$ as a function of the other relevant parameters in the problem. Use the average oil velocity $\overline{v}_o = Q_{oil} / \pi R_1^2$ and core radius R_1 as two of your primary variables together with as many other parameters as you need. Which dimensionless group is important in determining whether waves will develop. Based on your physical understanding of interfacial processes, express an appropriate inequality on the range for this dimensionless parameter in order for waves not to form.

b) Assuming that your criterion above is satisfied so that the flow in the pipe remains a perfect smooth core-annular flow as shown in the sketch, write down the appropriate boundary condition for the shear stress on the interface $r = R_1$. Furthermore, provide a criterion under which the change in pressure across the interface is negligible.

c) Use these boundary conditions to find expressions for the fully-developed velocity field $v_z(r)$ that are valid in the core domain $0 \le r \le R_1$ and the shell $R_1 \le r \le$ R. On a single large graph (at least 0.5 page in size), sketch the velocity profile **and** the shear stress profile across the entire pipe (i.e. for the region $0 \le r \le R$).

d) The flow in the pipeline is typically started impulsively by imposing a sudden increase in the pressure gradient along the pipe, and the flow takes a period of time to become fully developed. Draw a large diagram and sketch the shape of the velocity field $v_z(r,t)$ as a function of time. Provide an engineering estimate of the total time taken for the flow field to reach steady state.

e) Find expressions for the volume flow rate of oil Q_o and for the volume flow rate of water Q_w through the pipeline as a function of the imposed pressure ΔP and the other physical parameters defined in the figure.

f) The results of your analysis can be used to optimize the lubricated pipeline operation. For example; consider the viscosities μ_1, μ_2 and density ρ to all be held constant. Show that at any fixed value of the imposed pressure gradient $\Delta P/L$ there is an optimal value of the core radius (denoted R_1^*) that maximizes the volume flow rate of oil through the pipe. Derive an expression for R_1^* and explain (very briefly) why this occurs.

NOT REQUIRED FOR EXAM (extra credit):

g) If the outer layer of fluid becomes very thin ($\delta << R$) and of very low (but still non-zero!) viscosity ($\mu_2 << \mu_1$) then a number of simplifying approximations can be made in the governing equations. Show that in this limit the inner fluid can have a large velocity very close to the wall which makes it *appear* to 'slip' at the wall (i.e. at $r = R_1 \approx R$) with a slip velocity v_w that is proportional to the wall shear stress. Find the coefficient of proportionality.

Navier Stokes equation in $\{r, \theta, z\}$ coordinates:

$$\rho\left(\frac{\partial v_r}{\partial t} + v_r\frac{\partial v_r}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_r}{\partial \theta} - \frac{v_\theta^2}{r} + v_z\frac{\partial v_r}{\partial z}\right) = -\frac{\partial p}{\partial r} + \mu\left[\frac{\partial}{\partial r}\left(\frac{1}{r}\frac{\partial (r v_r)}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_r}{\partial \theta^2} + \frac{\partial^2 v_r}{\partial z^2} - \frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}\right] + \rho g_r$$

$$\rho\left(\frac{\partial v_z}{\partial t} + v_r\frac{\partial v_z}{\partial r} + \frac{v_\theta}{r}\frac{\partial v_z}{\partial \theta} + v_z\frac{\partial v_z}{\partial z}\right) = -\frac{\partial p}{\partial z} + \mu\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial v_z}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 v_z}{\partial \theta^2} + \frac{\partial^2 v_z}{\partial z^2}\right] + \rho g_z$$

the continuity equation is: $$\frac{1}{r}\frac{\partial (r v_r)}{\partial r} + \frac{1}{r}\frac{\partial v_\theta}{\partial \theta} + \frac{\partial v_z}{\partial z} = 0$$

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