LINEARIZATION OF FREE-SURFACE CONDITIONS

ON EARTH THE GRAVITATIONAL ACCELERATION IS LARGE ENOUGH THAT THE RESTORING ROLE IT PLAYS LEADS TO SMALL WAVE SLOPES IN MOST, BUT NOT ALL, CASES.

SO IT IS OFTEN A VERY GOOD ASSUMPTION TO SET

$$|\nabla \zeta| = O(\varepsilon)$$

WHERE ε IS A SMALL PARAMETER. THIS INVITES THE USE OF THE VERY POWERFUL TOOLS OF PERTURBATION THEORY.

LET:

$$\zeta = \underbrace{\zeta_1}_{O(\varepsilon)} + \underbrace{\zeta_2}_{O(\varepsilon^2)} + \underbrace{\zeta_3}_{O(\varepsilon^3)} + \cdots$$

$$\phi = \underbrace{\phi_1}_{\varepsilon} + \underbrace{\phi_2}_{\varepsilon^2} + \underbrace{\phi_3}_{\epsilon^3} + \cdots$$

AND DERIVE BOUNDARY VALUE PROBLEMS FOR (ζ_i, ϕ_i). RARELY WE NEED TO GO BEYOND $i=3$.

HERE WE WILL DERIVE THE FREE-SURFACE CONDITIONS UP TO SECOND ORDER.

THE MAIN TECHNIQUE IS TO EXPAND THE KINEMATIC AND DYNAMIC FREE SURFACE CONDITIONS ABOUT THE $z=0$ PLANE AND DERIVE STATEMENTS FOR THE UNKNOWN PAIRS (ϕ_1, ζ_1) AND (ϕ_2, ζ_2) AT $z=0$. LATER ON, THE SAME TECHNIQUE WILL BE USED TO LINEARIZE THE BODY BOUNDARY CONDITION AT $U=0$ (ZERO SPEED) AND $U>0$ (FORWARD SPEED).

KINEMATIC CONDITION

$$\left(\frac{\partial \zeta}{\partial t} + \nabla\phi \cdot \nabla\zeta \right)_{z=\zeta} = \left(\frac{\partial \phi}{\partial z} \right)_{z=\zeta}$$

$$\left(\frac{\partial \zeta}{\partial t} + \nabla\phi \cdot \nabla\zeta \right)_{z=0} + \zeta \frac{\partial}{\partial z}\left(\frac{\partial \zeta}{\partial t} + \nabla\phi \cdot \nabla\zeta \right)_{z=0} + \cdots$$

$$= \left(\frac{\partial \phi}{\partial z} \right)_{z=0} + \zeta \left(\frac{\partial^2 \phi}{\partial z^2} \right)_{z=0} + \cdots$$

INTRODUCE: $\left.\begin{aligned} \zeta &= \zeta_1 + \zeta_2 + \cdots \\ \phi &= \phi_1 + \phi_2 + \cdots \end{aligned}\right\}$ AND KEEP TERMS OF $O(\epsilon)$, $O(\epsilon^2)$, ...

DYNAMIC CONDITION

$$\zeta(x,y,t) = -\frac{1}{g}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right)_{z=\zeta}$$

$$\left.\begin{aligned}\zeta = &-\frac{1}{g}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right)_{z=0} \\ &-\frac{1}{g}\,\zeta\,\frac{\partial}{\partial z}\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right)_{z=0} + \cdots\end{aligned}\right\} \begin{aligned}\zeta &= \zeta_1 + \zeta_2 + \cdots \\ \phi &= \phi_1 + \phi_2 + \cdots\end{aligned}$$

LINEAR PROBLEM : $O(\varepsilon)$

$$\frac{\partial \zeta_1}{\partial t} = \frac{\partial \phi_1}{\partial z}, \quad z = 0; \quad \text{KINEMATIC}$$

$$\zeta_1 = -\frac{1}{g}\frac{\partial \phi_1}{\partial t}, \quad z=0; \quad \text{DYNAMIC}$$

PRESSURE FROM BERNOULLI, W. CONSTANT TERMS SET EQUAL TO ZERO, AT A FIXED POINT IN THE FLUID DOMAIN AT $\vec{x} = (X, Y, Z)$ IS GIVEN BY:

$$p = -\rho\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + gz\right); \quad \phi = \phi_1 + \phi_2 + \cdots$$

$$p = p_0 + p_1 + p_2$$

$$p_0 = -\rho g z \quad ; \quad \text{HYDROSTATIC}$$

$$p_1 = -\rho\, \partial\phi_1/\partial t \; ; \; \text{LINEAR}$$

ELIMINATING ζ_1 FROM THE KINEMATIC AND DYNAMIC FREE SURFACE CONDITIONS, WE OBTAIN THE CLASSICAL LINEAR FREE SURFACE CONDITION:

$$\begin{cases} \dfrac{\partial^2 \phi_1}{\partial t^2} + g \dfrac{\partial \phi_1}{\partial z} = 0, & z = 0 \\[2ex] \zeta_1 = -\dfrac{1}{g} \dfrac{\partial \phi_1}{\partial t}, & z = 0 \end{cases}$$

WITH: $\quad P_1 = -\rho \, \partial \phi_1 / \partial t$, AT SOME FIXED POINT $\vec{x}$

NOTE THAT ON $z=0$, $P_1 \neq 0$. IN FACT IT CAN BE OBTAINED FROM THE EXPRESSIONS ABOVE IN THE FORM

$$P_1 = \rho g \zeta_1 \quad , \quad z = 0.$$

SO LINEAR THEORY STATES THAT THE LINEAR PERTURBATION PRESSURE ON THE $z=0$ PLANE DUE TO A SURFACE WAVE DISTURBANCE IS EQUAL TO THE POSITIVE (NEGATIVE) "HYDROSTATIC" PRESSURE INDUCED BY THE POSITIVE (NEGATIVE) WAVE ELEVATION ζ_1. –

SECOND-ORDER PROBLEM : $O(\epsilon^2)$

- $$\frac{\partial \zeta_2}{\partial t} + \nabla\phi_1 \cdot \nabla\zeta_1 = \frac{\partial \phi_2}{\partial z} + \zeta_1 \frac{\partial^2 \phi_1}{\partial z^2}, \quad z=0$$

{ KINEMATIC CONDITION

- $$\zeta_2 = -\frac{1}{g}\left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\nabla\phi_1 \cdot \nabla\phi_1\right)_{z=0}$$
$$-\frac{1}{g}\zeta_1 \frac{\partial^2 \phi_1}{\partial z \partial t}, \quad z=0$$

{ DYNAMIC CONDITION

ALTERNATIVELY, THE KNOWN LINEAR TERMS MAY BE MOVED IN THE RIGHT-HAND SIDE AS FORCING FUNCTIONS, LEADING TO:

KINEMATIC SECOND-ORDER CONDITION

- $$\frac{\partial \zeta_2}{\partial t} - \frac{\partial \phi_2}{\partial z} = \zeta_1 \frac{\partial^2 \phi_1}{\partial z^2} - \underbrace{\nabla\phi_1 \cdot \nabla\zeta_1}_{\frac{\partial \phi_1}{\partial x}\frac{\partial \zeta_1}{\partial x} + \frac{\partial \phi_1}{\partial y}\frac{\partial \zeta_1}{\partial y}} \quad ; \; z=0$$

DYNAMIC SECOND-ORDER CONDITION

- $$\zeta_2 + \frac{1}{g}\frac{\partial \phi_2}{\partial t} = -\frac{1}{g}\left(\frac{1}{2}\nabla\phi_1 \cdot \nabla\phi_1 + \zeta_1 \frac{\partial^2 \phi_1}{\partial z \partial t}\right)_{z=0}$$

$$P_2 = -\rho\left(\frac{\partial \phi_2}{\partial t} + \frac{1}{2}\nabla\phi_1 \cdot \nabla\phi_1\right); \text{ AT } \vec{X}.$$

- THE VERY ATTRACTIVE FEATURE OF SECOND ORDER SURFACE WAVE THEORY IS THAT IT ALLOWS THE PRIOR SOLUTION OF THE LINEAR PROBLEM WHICH IS OFTEN POSSIBLE ANALYTICALLY AND NUMERICALLY.

- THE LINEAR SOLUTION IS THEN USED AS A FORCING FUNCTION FOR THE SOLUTION OF THE SECOND ORDER PROBLEM. THIS IS OFTEN POSSIBLE ANALYTICALLY AND IN MOST CASES NUMERICALLY IN THE ABSENCE OR PRESENCE OF BODIES.

- LINEAR AND SECOND-ORDER THEORIES ARE VERY APPROPRIATE TO USE FOR THE MODELING OF SURFACE WAVES AS STOCHASTIC PROCESSES.

- BOTH THEORIES ARE VERY USEFUL IN PRACTICE AS WILL BE DEMONSTRATED IN MANY CONTEXTS IN THE PRESENT COURSE, PARTICULARLY IN CONNECTION WITH WAVE-BODY INTERACTIONS.—