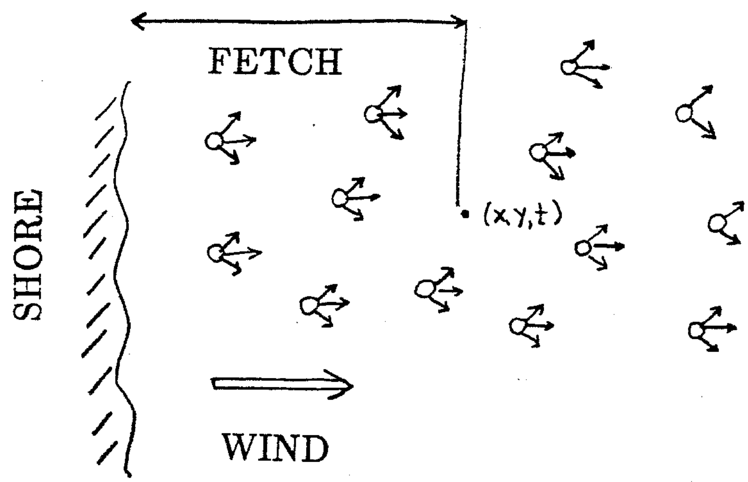
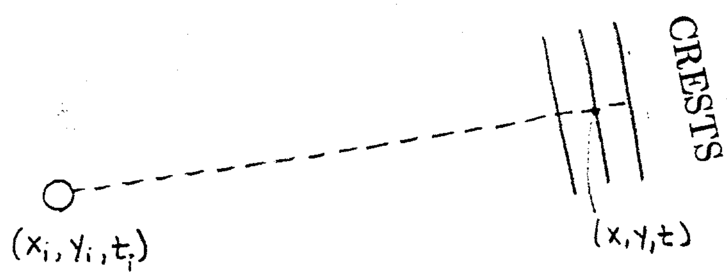
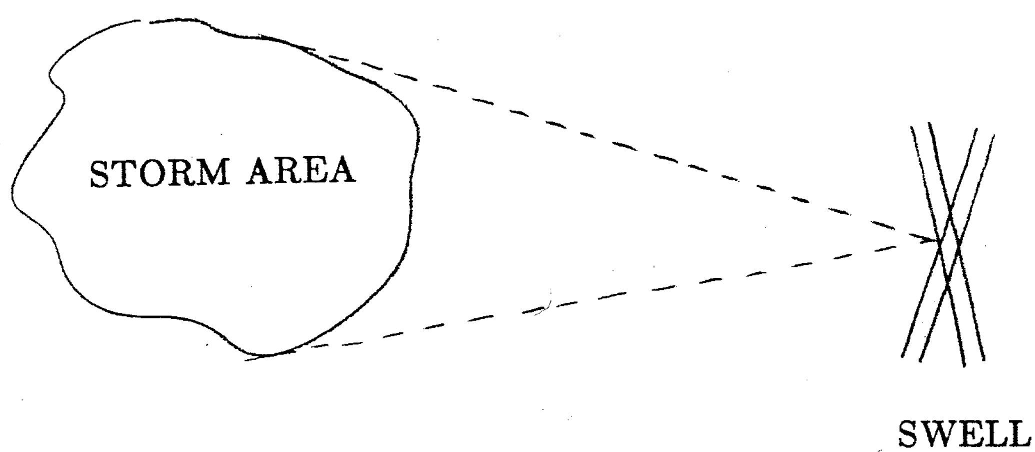
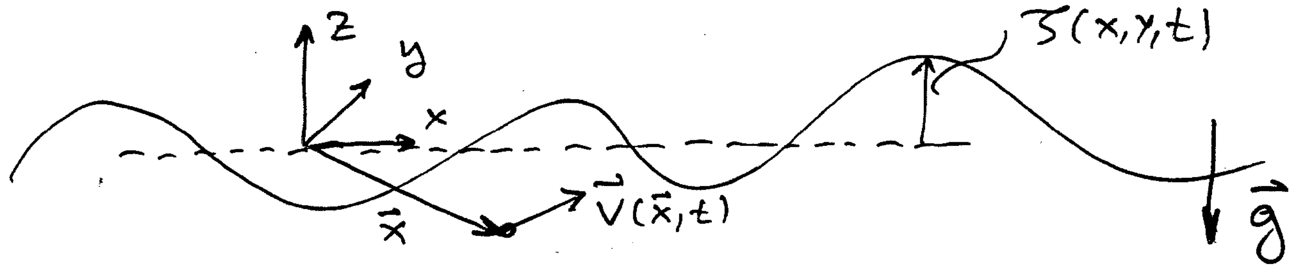
THE OCEAN ENVIRONMENT







NONLINEAR FREE-SURFACE CONDITION



- (x,y,z) : EARTH FIXED COORDINATE SYSTEM

  $\vec{x}$: FIXED EULERIAN VECTOR

  $\vec{V}$: FLOW VELOCITY VECTOR AT $\vec{x}$

  ζ : FREE SURFACE ELEVATION

- ASSUME IDEAL FLUID (NO SHEAR STRESSES) AND IRROTATIONAL FLOW:

$$\nabla \times \vec{V} = 0$$

LET :

$$\vec{V} = \nabla \phi \quad \Rightarrow \quad \nabla \times \nabla \phi \equiv 0$$

WHERE $\phi(\vec{x},t)$ IS THE VELOCITY POTENTIAL ASSUMED SUFFICIENTLY CONTINUOUSLY DIFFERENTIABLE

- POTENTIAL FLOW MODEL OF SURFACE WAVE PROPAGATION AND WAVE-BODY INTERACTIONS VERY ACCURATE. FEW IMPORTANT EXCEPTIONS WILL BE NOTED

- CONSERVATION OF MASS:

$$\nabla \cdot \vec{V} = 0 \Rightarrow$$

$$\nabla \cdot \nabla \phi = 0 \Rightarrow \nabla^2 \phi = 0 \quad \text{OR}$$

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} = 0, \quad \text{LAPLACE EQUATION}$$

- CONSERVATION OF LINEAR MOMENTUM.

EULER'S EQUATION IN THE ABSENCE OF VISCOSITY

$$\frac{\partial \vec{V}}{\partial t} + (\vec{V} \cdot \nabla) \vec{V} = -\frac{1}{\rho} \nabla p + \vec{g}$$

$p(\vec{x}, t)$: FLUID PRESSURE AT $(\vec{x}, t)$

$\vec{g} = -\vec{k} g$: ACCELERATION OF GRAVITY

$\vec{k}$: UNIT VECTOR POINTING IN THE POSITIVE Z-DIRECTION

ρ: WATER DENSITY

- VECTOR IDENTITY:

$$(\vec{V} \cdot \nabla) \vec{V} = \frac{1}{2} \nabla (\vec{V} \cdot \vec{V}) - \vec{V} \times (\nabla \times \vec{V})$$

IN IRROTATIONAL FLOW: $\nabla \times \vec{V} = 0$, THUS

EULER'S EQUATIONS BECOME:

$$\frac{\partial \vec{V}}{\partial t} + \frac{1}{2}\nabla(\vec{V}\cdot\vec{V}) = -\frac{1}{\rho}\nabla p - \nabla(gz)$$

NOTE: $\nabla z = \vec{k}$, $\vec{V} = \nabla\phi$

UPON SUBSTITUTION:

$$\nabla\Big(\underbrace{\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + p/\rho + gz}_{F(\vec{x},t)}\Big) = 0$$

$$\Rightarrow \quad \nabla F(\vec{x},t) = 0 \quad \Rightarrow \quad F(\vec{x},t) = \underset{\text{CONSTANT}}{\overset{\|}{\mathbb{C}}}$$

BERNOULLI'S EQUATION FOLLOWS:

$$\frac{\partial\phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + p/\rho + gz = \mathbb{C}$$

OR

$$p/\rho = -\frac{\partial\phi}{\partial t} - \frac{1}{2}\nabla\phi\cdot\nabla\phi - gz + \mathbb{C}$$

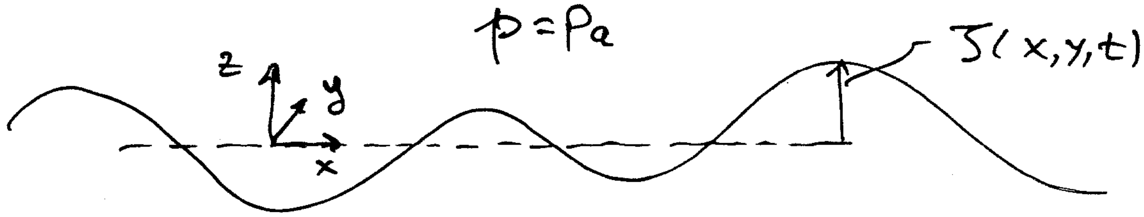
THE VALUE OF THE CONSTANT $\mathbb{C}$ IS IMMATERIAL AS WILL BE SHOWN BELOW.

- ANGULAR MOMENTUM CONSERVATION PRINCIPLE CONTAINED IN:

$$\nabla\times\vec{V} = 0$$

- IF PARTICLES ARE MODELED AS SPHERES, ABOVE EQUATION IMPLIES NO ANGULAR VELOCITY AT ALL TIMES

DERIVATION OF NONLINEAR FREE-SURFACE CONDITION



- **METHOD I:** ON $z = \zeta$; $p = p_a \equiv$ ATMOSPHERIC PRESSURE

FROM BERNOULLI:

$$\rightarrow \quad p_a/\rho = -\frac{\partial \phi}{\partial t} - \frac{1}{2}\nabla\phi\cdot\nabla\phi - g\zeta + C$$

on $z = \zeta(x,y,t)$

ON $z = \zeta$ THE MATHEMATICAL FUNCTION

$$z - \zeta(x,y,t) \equiv \mathcal{F}(x,y,z,t)$$

IS ALWAYS ZERO WHEN TRACING A FLUID PARTICLE ON THE FREE SURFACE. SO THE SUBSTANTIAL OR TOTAL DERIVATIVE OF $\mathcal{F}$ MUST VANISH, THUS

$$\frac{D\mathcal{F}}{Dt} = 0 = \left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla\right)\mathcal{F} = 0, \text{ on } z = \zeta$$

EXPANDING WE OBTAIN:

$$\left(\frac{\partial}{\partial t} + \vec{V} \cdot \nabla \right) (z - \zeta) = 0, \quad \text{on} \quad z = \zeta$$

$$\rightarrow \quad \frac{\partial \zeta}{\partial t} + \frac{\partial \phi}{\partial x} \frac{\partial \zeta}{\partial x} + \frac{\partial \phi}{\partial y} \frac{\partial \zeta}{\partial y} = \frac{\partial \phi}{\partial z} \, , \quad z = \zeta$$

{ KINEMATIC FREE-SURFACE CONDITION

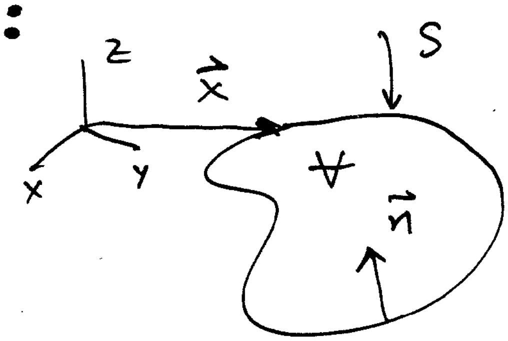
FROM BERNOULLI'S EQUATION WE OBTAIN THE DYNAMIC FREE SURFACE CONDITION:

$$\rightarrow \quad \frac{\partial \phi}{\partial t} + \frac{1}{2} \nabla \phi \cdot \nabla \phi + g \zeta = C - P_a / \rho \, , \quad z = \zeta$$

{ DYNAMIC FREE-SURFACE CONDITION

CONSTANTS IN BERNOULLI'S EQUATION MAY BE SET EQUAL TO ZERO WHEN WE ARE EVENTUALLY INTERESTED IN INTEGRATING PRESSURES OVER CLOSED OR OPEN BOUNDARIES (FLOATING OR SUBMERGED BODIES) TO OBTAIN FORCES & MOMENTS. THIS FOLLOWS FROM A SIMPLE APPLICATION OF ONE OF THE TWO GAUSS VECTOR THEOREMS WE WILL USE A LOT IN THIS COURSE:

GAUSS I:



$\vec{n}$: UNIT NORMAL VECTOR POINTING INSIDE THE VOLUME $\forall$

$f(\bar{x})$: ARBITRARY SUFFICIENTLY DIFFERENTIABLE SCALAR FUNCTION

VECTOR IDENTITY:
$$\iiint_{\forall} \nabla f \, dv = -\iint_S f \vec{n} \, ds$$

NOTE THE THREE SCALAR IDENTITIES THAT FOLLOW:

$$\iiint_{\forall} \frac{\partial f}{\partial x} \, dv = -\iint_S f n_1 \, ds$$

$$\iiint_{\forall} \frac{\partial f}{\partial y} \, dv = -\iint_S f n_2 \, ds$$

$$\iiint_{\forall} \frac{\partial f}{\partial z} \, dv = -\iint_S f n_3 \, ds \, .-$$

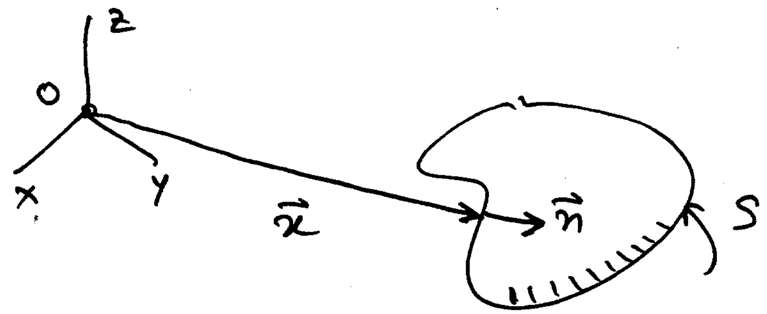
GAUSS II:

$\vec{V}$: ARBITRARY SUFFICIENTLY DIFFERENTIABLE VECTOR FUNCTION

SCALAR IDENTITY:
$$\iiint_{\forall} \nabla \cdot \vec{V} = -\iint_S \vec{V} \cdot \vec{n} \, ds$$

SCALAR IDENTITY OFTEN USED TO PROVE MASS CONSERVATION PRINCIPLE

DEFINITION OF FORCE & MOMENT IN TERMS OF FLUID PRESSURE



$$\vec{F} = \iint_S p\,\vec{n}\,ds$$

$$\vec{M} = \iint_S p\,(\vec{x} \times \vec{n})\,ds$$

IT FOLLOWS FROM GAUSS I THAT IF $p = C$ THE FORCE AND MOMENT OVER A CLOSED BOUNDARY S VANISH IDENTICALLY. HENCE WITHOUT LOSS OF GENERALITY IN THE CONTEXT OF WAVE BODY INTERACTIONS WE WILL SET $C = 0$.

IT FOLLOWS THAT THE DYNAMIC FREE SURFACE CONDITION TAKES THE FORM

$$\zeta(x,y,t) = -\frac{1}{g}\left\{\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi\right\}_{z=\zeta} \quad \bullet-$$

METHOD II :

WHEN TRACING A FLUID PARTICLE ON THE FREE SURFACE THE HYDRODYNAMIC PRESSURE GIVEN BY BERNOULLI (AFTER THE CONSTANT $\mathbb{C}$ HAS BEEN SET EQUAL TO ZERO) MUST VANISH AS WE FOLLOW THE PARTICLE:

$$\frac{D}{Dt}\left\{ \frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + gz \right\} = 0, \qquad z = \zeta$$

OR

$$\left(\frac{\partial}{\partial t} + \vec{V}\cdot\nabla \right)\left(\frac{\partial \phi}{\partial t} + \frac{1}{2}\nabla\phi\cdot\nabla\phi + gz \right) = 0$$
$$z = \zeta$$

THIS CONDITION ALSO FOLLOWS UPON ELIMINATION OF ζ FROM THE KINEMATIC & DYNAMIC CONDITIONS DERIVED UNDER METHOD I.

THIS COMPLETES THE STATEMENT OF THE NONLINEAR BOUNDARY VALUE PROBLEM SATISFIED BY SURFACE WAVES OF LARGE AMPLITUDE IN POTENTIAL FLOW AND IN THE ABSENCE OF WAVE BREAKING.