**2.20 - Marine Hydrodynamics**
**Lecture 6**

## 2.2 Similarity Parameters from Governing Equations and Boundary Conditions

In this paragraph we will see how we can specify the SP's for a problem that is governed by the Navier-Stokes equations. The SP's are obtained by **scaling**, **non-dimensionalizing** and **normalizing** the governing equations and boundary conditions.

1. **Scaling** First step is to identify the **characteristic scales** of the problem.

   For example: Assume a flow where the velocity magnitude at any point in space or time $|\vec{v}(\vec{x},t)|$ is *about equal* to a velocity $U$, i.e. $|\vec{v}(\vec{x},t)| = \alpha\, U$, where $\alpha$ is such that $0 \leq \alpha \sim O(1)$. Then $U$ can be chosen to be the **characteristic velocity** of the flow and any velocity $\vec{v}$ can be written as:

   $$\vec{v} = U\vec{v}^\star$$

   where it is evident that $\vec{v}^\star$ is:

   (a) **dimensionless** (no units), and
   (b) **normalized** ($|\vec{v}^\star| \sim O(1)$).

   Similarly we can specify characteristic length, time, pressure etc scales:

| Characteristic scale | | Dimensionless and normalized quantity | Dimensional quantity in terms of characteristic scale |
|---|---|---|---|
| Velocity | $U$ | $\vec{v}^\star$ | $\vec{v} = U\vec{v}^\star$ |
| Length | $L$ | $\vec{x}^\star$ | $\vec{x} = L\vec{x}^\star$ |
| Time | $T$ | $t^\star$ | $t = Tt^\star$ |
| Pressure | $p_o - p_v$ | $p^\star$ | $p = (p_o - p_v)p^\star$ |

2. **Non-dimensionalizing and normalizing the governing equations and boundary conditions**

   Substitute the dimensional quantities with their non-dimensional expressions (eg. substitute $\vec{v}$ with $U\vec{v}^\star$, $\vec{x}$ with $L\vec{x}^\star$, etc) into the governing equations, and boundary conditions. The linearly independent, non-dimensional ratios between the characteristic quantities (eg. $U$, $L$, $T$, $p_o - p_v$) are the SP's.

   (a) Substitute into the Continuity equation (incompressible flow)

$$\begin{aligned}
\nabla \cdot \vec{v} &= 0 \Rightarrow \\
\frac{U}{L}\nabla^\star \cdot \vec{v}^\star &= 0 \Rightarrow \\
\nabla^\star \cdot \vec{v}^\star &= 0
\end{aligned}$$

   Where all the $()^\star$ quantities are **dimensionless** and **normalized** (i.e., O(1)), for example, $\frac{\partial \vec{v}^\star}{\partial x^\star} = O(1)$.

   (b) Substitute into the Navier-Stokes (momentum) equations

$$\begin{aligned}
\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\,\vec{v} &= -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v} - g\hat{j} \Rightarrow \\
\frac{U}{T}\frac{\partial \vec{v}^\star}{\partial t^\star} + \frac{U^2}{L}\left(\vec{v}^\star \cdot \nabla^\star\right)\vec{v}^\star &= -\frac{p_o - p_v}{\rho U^2}\nabla^\star p^\star + \frac{\nu U}{L^2}(\nabla^\star)^2 \vec{v}^\star - g\hat{j}
\end{aligned}$$

   divide through by $\frac{U^2}{L}$, i.e., order of magnitude of the convective inertia term $\Rightarrow$

$$\widetilde{\frac{L}{UT}}\left(\frac{\partial \vec{v}}{\partial t}\right)^\star + \Big((\vec{v} \cdot \nabla)\vec{v}\Big)^\star = -\widetilde{\frac{p_o - p_v}{\rho U^2}}\,(\nabla p)^\star + \widetilde{\frac{\nu}{UL}}\,(\nabla^2 \vec{v})^\star - \widetilde{\frac{gL}{U^2}}\hat{j}$$

   The coefficients ( $\widetilde{\quad}$ ) are SP's.

Since all the dimensionless and normalized terms $()^{\star}$ are of O(1), the SP's ( $\widetilde{\qquad}$ ) measure the relative importance of each term compared to the convective inertia. Namely,

- $\dfrac{L}{UT} \equiv S =$ Strouhal number $\sim \dfrac{\text{Eulerian inertia}}{\text{convective inertia}} \sim \dfrac{\frac{\partial \vec{v}}{\partial t}}{(\vec{v} \cdot \nabla)\vec{v}}$

  The Strouhal number $S$ is a measure of transient behavior.

  For example assume a ship of length $L$ that has been travelling with velocity $U$ for time $T$. If the $T$ is much larger than the time required to travel a ship length, then we can assume that the ship has reached a steady-state.

$$
\begin{aligned}
\frac{L}{U} \quad &<< \quad T \Rightarrow \\
\frac{L}{UT} = S \quad &<< \quad 1 \Rightarrow \\
\text{ignore } \frac{\partial \vec{v}}{\partial t} \quad &\rightarrow \quad \text{assume steady-state}
\end{aligned}
$$

- $\dfrac{p_o - p_v}{\frac{1}{2}\rho U^2} \equiv \sigma =$ cavitation number.

  The cavitation number $\sigma$ is a measure of the likelihood of cavitation.

  If $\sigma >> 1$, no cavitation. If cavitation is not a concern we can choose $p_o$ as a characteristic pressure scale, and non-dimensionalize the pressure $p$ as $p = p_o p^{\star}$

- $\dfrac{p_o}{\frac{1}{2}\rho U^2} \equiv E_u =$ Euler number $\sim \dfrac{\text{pressure force}}{\text{inertia force}}$

- $\dfrac{UL}{\nu} \equiv R_e =$ Reynold's number $\sim \dfrac{\text{inertia force}}{\text{viscous force}}$

  If $R_e >> 1$, ignore viscosity.

- $\sqrt{\dfrac{U^2}{gL}} = \dfrac{U}{\sqrt{gL}} \equiv F_r =$ Froude number $\sim \left(\dfrac{\text{inertia force}}{\text{gravity force}}\right)^{\frac{1}{2}}$

(c) Substitute into the kinematic boundary conditions

$$\begin{aligned} \vec{u} &= \vec{U}_{boundary} \Rightarrow \\ \vec{u}^{\star} &= \vec{U}^{\star}_{boundary} \end{aligned}$$

(d) Substitute into the dynamic boundary conditions

$$\begin{aligned} p &= p_a + \Delta p = p_a + \underbrace{\left(\frac{1}{R_1} + \frac{1}{R_2}\right)\sum}_{\Delta p} \quad \overset{R=LR^{\star}}{\underset{p=(p_o-p_v)p^{\star}}{\Longrightarrow}} \\ p^{\star} &= p_a^{\star} + \frac{\sum}{(p_o - p_v)\,L}\left(\frac{1}{R_1^{\star}} + \frac{1}{R_2^{\star}}\right) = p_a^{\star} + \frac{2}{\sigma}\left(\frac{\widetilde{\sum/\rho}}{U^2 L}\right) \end{aligned}$$

- $\dfrac{U^2 L}{\sum/\rho} \equiv W_e =$ Weber number $\sim \dfrac{\text{inertial forces}}{\text{surface tension forces}}$

- Some SP's used in hydrodynamics (the table is *not* exhaustive):

| SP | | Definition | | |
|---|---|---|---|---|
| Reynold's number | $R_e$ | $\frac{UL}{\nu}$ | $\sim$ | $\frac{\text{inertia}}{\text{viscous}}$ |
| Froude number | $F_r$ | $\sqrt{\frac{U^2}{gL}}$ | $\sim$ | $\frac{\text{inertia}}{\text{gravity}}$ |
| Euler number | $E_u$ | $\frac{p_o}{\frac{1}{2}\rho U^2}$ | $\sim$ | $\frac{\text{pressure}}{\text{inertia}}$ |
| Cavitation number | $\sigma$ | $\frac{p_o-p_v}{\frac{1}{2}\rho U^2}$ | $\sim$ | $\frac{\text{pressure}}{\text{inertia}}$ |
| Strouhal number | $S$ | $\frac{L}{UT}$ | $\sim$ | $\frac{\text{Eulerian inertia}}{\text{convective inertia}}$ |
| Weber number | $W_e$ | $\frac{U^2 L}{\Sigma/\rho}$ | $\sim$ | $\frac{\text{inertia}}{\text{surface tension}}$ |

## 2.3 Similarity Parameters from Physical Arguments

Alternatively, we can obtain *the same SP's* by taking the dimensionless ratios of significant flow quantities. Physical arguments are used to identify the significant flow quantities. Here we obtain SP's from force ratios. We first identify the types of dominant forces acting on the fluid particles. The SP's are merely the ratios of those forces.

1. Identify the type of forces that act on a fluid particle:

   1.1 Inertial forces $\sim$ mass $\times$ acceleration $\sim (\rho L^3)\left(\frac{U^2}{L}\right) = \rho U^2 L^2$

   1.2 Viscous forces $\sim \underbrace{\mu \dfrac{\partial u}{\partial y}}_{\text{shear stress}}$ $\times$ area $\sim \left(\mu \frac{U}{L}\right)(L^2) = \mu U L$

   1.3 Gravitational forces $\sim$ mass $\times$ gravity $\sim (\rho L^3) g$

   1.4 Pressure forces $\sim (p_o - p_v) L^2$

2. For similar streamlines, particles must be acted on forces whose resultants are in the same direction at geosimilar points. Therefore, the following force **ratios** must be equal:

   - $\dfrac{\text{inertia}}{\text{viscous}} \sim \dfrac{\rho U^2 L^2}{\mu U L} = \dfrac{UL}{\nu} \equiv R_e$

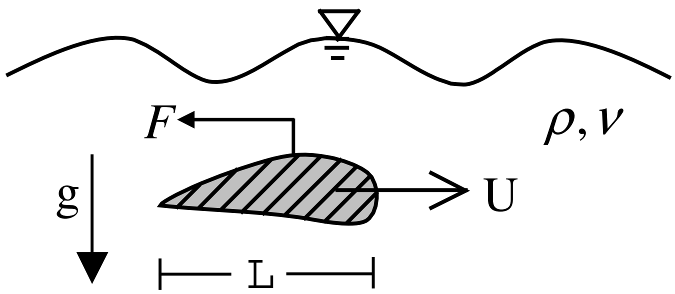
   - $\left(\dfrac{\text{inertia}}{\text{gravity}}\right)^{1/2} \sim \left(\dfrac{\rho U^2 L^2}{\rho g L^3}\right)^{1/2} = \dfrac{U}{\sqrt{gL}} \equiv F_r$

   - $\left(\dfrac{\frac{1}{2}\text{inertia}}{\text{pressure}}\right)^{-1} \sim \dfrac{(p_o - p_v)L^2}{\frac{1}{2}\rho U^2 L^2} = \dfrac{p_o - p_v}{\frac{1}{2}\rho U^2} \equiv \sigma$

## 2.4 Importance of SP's

- The SP's indicate whether different systems have *similar* flow properties.
- The SP's provide guidance in approximating complex physical problems.

**Example** A hydrofoil of length $L$ is submerged in a known fluid (density $\rho$, kinematic viscosity $\nu$). Given that the hydrofoil is travelling with velocity $U$ and the gravitational acceleration is $g$, determine the *hydrodynamic* force $F$ on the hydrofoil.




SP's for this problem:

$$\boxed{S = \frac{L}{UT}, \quad \sigma = \frac{p_o - p_v}{\frac{1}{2}\rho U^2}, \quad W_e = \frac{U^2 L}{\sum/\rho}, \quad F_r = \frac{U}{\sqrt{gL}}, \quad R_e = \frac{UL}{\nu}}$$

We define the dimensionless force coefficient:

$$C_F \equiv \frac{F}{\frac{1}{2}\rho U^2 L^2}$$

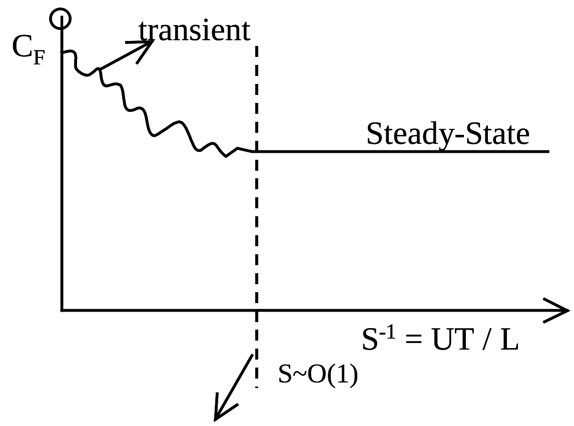
The force coefficient must depend on the other SP's:

$$\boxed{C_F = C_F'\left(S, \sigma, W_e, F_r, R_e\right) \quad \text{or} \quad C_F = C_F\left(S, \sigma^{-1}, W_e^{-1}, F_r, R_e^{-1}\right)}$$

**Procedure** We will first study under what conditions each SP $\to 0$.
We will estimate $C_F$ for the case that all of the SP's $\to 0$.

1. Significance of the Strouhal number $S = L/UT$.

Change $S$ keeping all other SP's ($\sigma^{-1}$, $W_e^{-1}$, $F_r$, $R_e^{-1}$) fixed.




For $S << 1$, assume steady-state: $\frac{\partial}{\partial t} = 0$
For $S >> 1$, unsteady effect is dominant.

For example, for the case $L = 10$m and $U = 10$m/s we can neglect the unsteady effects when:

$$S << 1 \qquad \Rightarrow \qquad \frac{L}{UT} << 1 \qquad \Rightarrow \qquad T >> \frac{L}{U} \qquad \Rightarrow \qquad T >> 1\text{s}$$

Therefore for $T >> 1$s we can approximate $S \simeq 1$ and we can assume steady state. In the case of a **steady** flow:

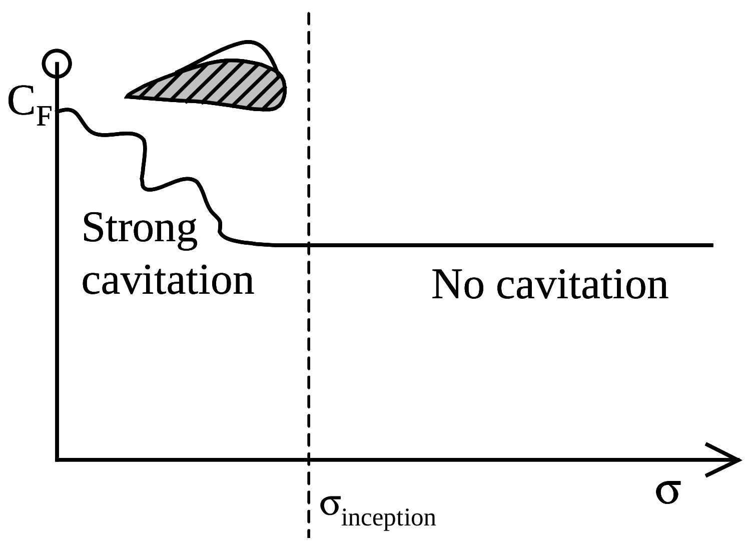
$$\begin{aligned} C_F &= C_F\left(S \simeq 0, \sigma^{-1}, W_e^{-1}, F_r, R_e^{-1}\right) \Rightarrow \\ C_F &\cong C_F\left(\sigma^{-1}, W_e^{-1}, F_r, R_e^{-1}\right) \end{aligned}$$

2. Significance of the cavitation number $\sigma = \dfrac{p_o - p_v}{\frac{1}{2}\rho U^2}$ .

Change $\sigma^{-1}$ keeping all other SP's ($S \simeq 0$, $W_e^{-1}$, $F_r$, $R_e^{-1}$) fixed.

*Some comments on cavitation*

| | | |
|---|---|---|
| $p_v$ | : | Vapor pressure, the pressure at which water boils |
| $p_o \leq p_v$ | : | State of fluid changes from liquid to gas $\Rightarrow$ CAVITATION |
| Consequences | : | Unsteady $\rightarrow$ Vibration of structures, which may lead to fatigue, etc |
| | | Unstable $\rightarrow$ Sudden cavity collapses |
| | | $\rightarrow$ Large force acting on the structure surface |
| | | $\rightarrow$ Surface erosion |




For $\sigma << 1$, cavitation occurs.

For $\sigma >> 1 \Rightarrow \sigma^{-1} << 1$, cavitation will *not* occur.

In general cavitation occurs when we have large velocities, or when $p_o \sim p_v$

For example, assume a hydrofoil travelling in water of density $\rho = 10^3 \mathrm{kg/m^3}$.

The characteristic pressure is $p_o = 10^5 \mathrm{N/m^2}$ and the vapor pressure is $p_v = 10^3 \mathrm{N/m^2}$. Cavitation will not occur when:

$$\sigma^{-1} << 1 \quad \Rightarrow \quad \frac{\frac{1}{2}\rho U^2}{p_o - p_v} << 1 \quad \Rightarrow \quad U << \sqrt{\frac{p_o - p_v}{\frac{1}{2}\rho}} \quad \Rightarrow \quad U << 14 \mathrm{m/s}$$

Therefore for $U << 14$m/s it is $\sigma >> 1 \Rightarrow \sigma^{-1} \simeq 0$ and cavitation will not occur.

In the case of a **steady**, **non-cavitating** flow:

$$\begin{aligned} C_F &= C_F\left(0, \sigma^{-1} \simeq 0, W_e^{-1}, F_r, R_e^{-1}\right) \Rightarrow \\ C_F &\cong C_F\left(W_e^{-1}, F_r, R_e^{-1}\right) \end{aligned}$$

3. Significance of the Weber number $W_e = \dfrac{U^2 L}{\Sigma/\rho}$.

   Change $W_e^{-1}$ keeping the other SP's ($S \simeq 0$, $\sigma^{-1} \simeq 0$, $F_r$, $R_e^{-1}$) fixed.

   For $W_e << 1$, surface tension is significant.
   For $W_e >> 1 \Rightarrow W_e^{-1} << 1$, surface tension is *not* significant.

   For example, assume a hydrofoil travelling with velocity $U = 1$m/s near an air/water interface (water density $\rho = 10^3$kg/m$^3$, surface tension coefficient $\sum = 0.07$N/m).

   Surface tension can be neglected when:

   $$W_e^{-1} << 1 \qquad \Rightarrow \qquad \frac{\frac{\Sigma}{\rho}}{U^2 L} << 1 \qquad \Rightarrow \qquad L >> \frac{\frac{\Sigma}{\rho}}{U^2} \qquad \Rightarrow \qquad L >> 7 \cdot 10^{-5} \text{ m}$$

   Therefore for $L >> 7 \cdot 10^{-5}$ m it is $W_e >> 1 W_e^{-1} \simeq 1$ and surface tension effects can be neglected.

   So in the case of a **steady**, **non-cavitating**, **non-surface tension** flow:

   $$\begin{aligned} C_F &= C_F\left(0, 0, W_e^{-1} \simeq 0, F_r, R_e^{-1}\right) \Rightarrow \\ C_F &\cong C_F\left(F_r, R_e^{-1}\right) \end{aligned}$$

4. Significance of the Froude number $F_r = \dfrac{U}{\sqrt{gh}}$, which measures the 'gravity effects'.

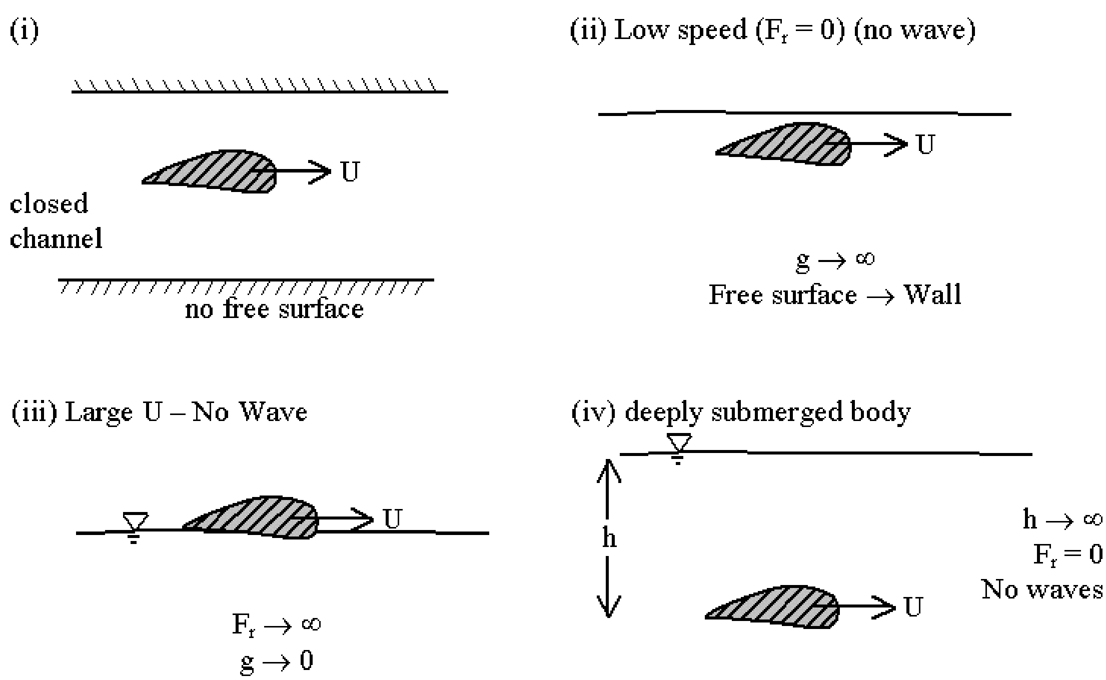
Change $F_r$ keeping the other SP's ($S \simeq 0$, $\sigma^{-1} \simeq 0$, $W_e^{-1} \simeq 0$, $R_e^{-1}$) fixed.

'Gravity effects', hydrostatic pressure do not create any flow (isotropic) nor do they change the flow dynamics **unless** Dynamic Boundary Conditions apply.

'Gravity effects' are not significant when $U << \sqrt{gh} \Rightarrow F_r \simeq 0$, or $U >> \sqrt{gh} \Rightarrow F_r^{-1} \simeq 0$. Physically, this is the case when the free surface is

- absent or
- far away or
- not disturbed, i.e., no wave generation.

The following figures (i - iv) illustrate cases where gravity effects are not significant.



In any of those cases the gravity effects are insignificant and equivalently $F_r$ is not important (i.e. $F_r \simeq 0$ or $F_r^{-1} \simeq 0$).

So in the case of a **steady**, **non-cavitating**, **non-surface tension**, with **no gravity effects** flow:

$$\begin{aligned} C_F &= C_F\left(0, 0, 0, F_r \simeq 0 \text{ or } F_r^{-1} \simeq 0, R_e^{-1}\right) \Rightarrow \\ C_F &\cong C_F\left(R_e^{-1}\right) \end{aligned}$$

*A look ahead: Froude's Hypothesis*

Froude's Hypothesis states that

$$C_F = C_F\left(F_r, R_e\right) = C_1\left(F_r\right) + C_2\left(R_e\right)$$

Therefore *dynamic similarity* requires

$$\begin{aligned} (R_e)_1 &= (R_e)_2, \text{ and} \\ (F_r)_1 &= (F_r)_2 \end{aligned}$$

**Example:** Show that if $\nu$ and $g$ are kept constant, two systems (1, 2) can be both *geometrically* and *dynamically similar* only if:

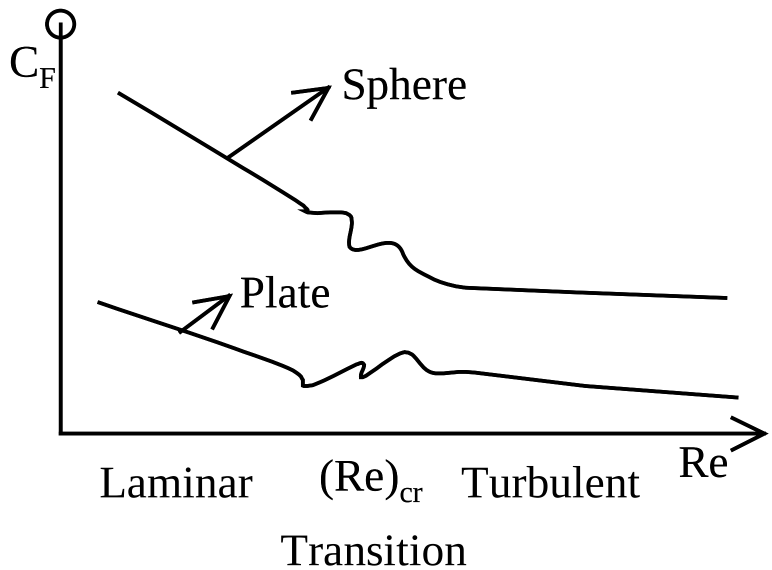
$$\begin{aligned} L_1 &= L_2, \text{ and} \\ U_1 &= U_2 \end{aligned}$$

5. Significance of the Reynolds number $R_e = \dfrac{UL}{\nu}$.

Change $R_e$ keeping the other SP's ($S \simeq 0$, $\sigma^{-1} \simeq 0$, $W_e^{-1} \simeq 0$, $F_r \simeq 0$ or $F_r^{-1} \simeq 0$) fixed.

Recall that for a **steady**, **non-cavitating**, **non-surface tension**, with **no gravity effects** flow:

$$C_F = C_F\left(R_e^{-1}\right)$$




| | |
|---|---|
| $R_e << 1$, | Stokes flow (creeping flow) |
| $R_e < (R_e)_{cr}$, | Laminar flow |
| $R_e > (R_e)_{cr}$, | Turbulent flow |
| $R_e \to \infty$, | Ideal fluid |

For example, a hydrofoil of cord length $L = 1m$ travelling in water (kinematic viscosity $\nu = 10^{-1}\mathrm{m}^2/\mathrm{s}$) with velocity $U = 10\mathrm{m/s}$ has a Reynolds number *with respect to $L$*:

$$R_e = \frac{\nu}{UL} = 10^7 \to \textbf{ideal fluid}\text{, and } R_e^{-1} \simeq 0$$

Therefore for a **steady**, **non-cavitating**, **non-surface tension**, with **no-gravity effects** flow in an **ideal fluid**:

$$C_F = C_F\,(0, 0, 0, 0, 0) = \text{constant} = 0$$

$\rightarrow$D'Alembert's Paradox

No drag force on moving body in ideal fluid.