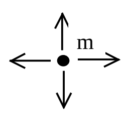
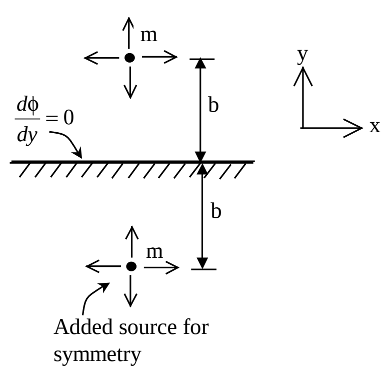
2.20 - Marine Hydrodynamics
Lecture 11

3.11 - Method of Images

- **Potential for single source:** $\phi = \frac{m}{2\pi} \ln \sqrt{x^2 + y^2}$

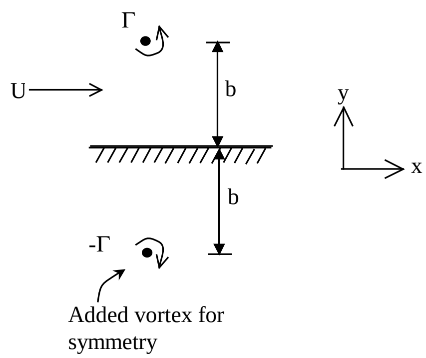


- **Potential for source near a wall:** $\phi = \frac{m}{2\pi} \left(\ln \sqrt{x^2 + (y-b)^2} + \ln \sqrt{x^2 + (y+b)^2} \right)$



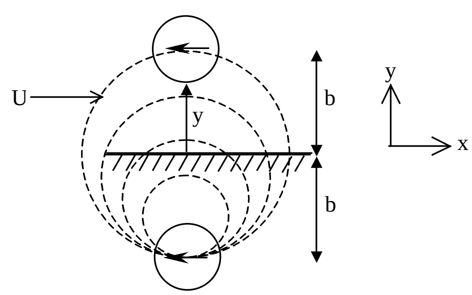
Note: Be sure to verify that the boundary conditions are satisfied by symmetry or by calculus for $\phi(y) = \phi(-y)$.

- **Vortex near a wall (ground effect):** $\phi = U_x + \frac{\Gamma}{2\pi}\left(\tan^{-1}(\frac{y-b}{x}) - \tan^{-1}(\frac{y+b}{x})\right)$



Verify that $\frac{d\phi}{dy} = 0$ on the wall $y = 0$.

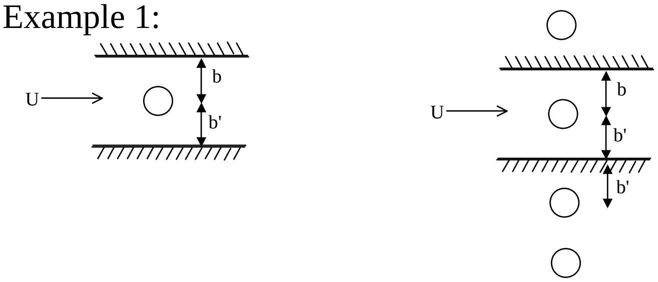
- **Circle of radius a near a wall:** $\phi \cong Ux\left(1 + \frac{a^2}{x^2 + (y-b)^2} + \frac{a^2}{x^2 + (y+b)^2}\right)$



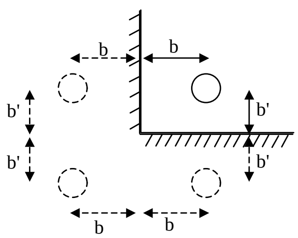
This solution satisfies the boundary condition on the wall ($\frac{\partial\phi}{\partial n} = 0$), and the degree it satisfies the boundary condition of no flux through the circle boundary increases as the ratio $b/a >> 1$, i.e., the velocity due to the image dipole small on the real circle for $b >> a$. For a 2D dipole, $\phi \sim \frac{1}{d}, \nabla\phi \sim \frac{1}{d^2}$.

- **More than one wall:**

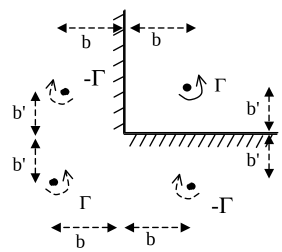
Example 1:



Example 2:

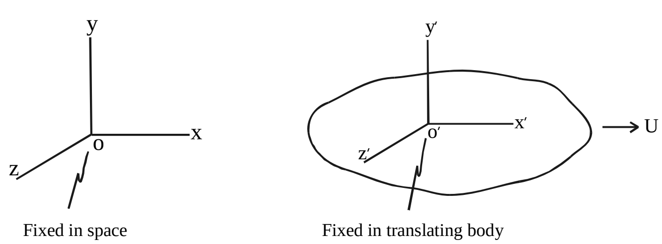


Example 3:

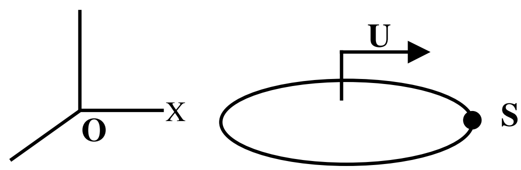
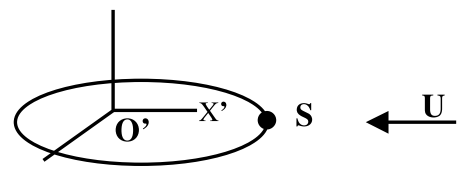


3.12 Forces on a body undergoing steady translation "D'Alembert's paradox"

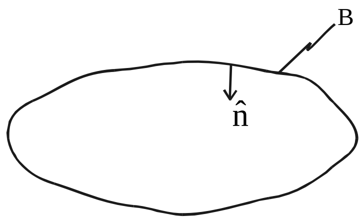
3.12.1 Fixed bodies & translating bodies - Galilean transformation.



$$x = x` + Ut$$

<table>
<tr><td>Reference system O: $\vec{v}, \phi, p$</td><td>Reference system O': $\vec{v}', \phi', p'$</td></tr>
<tr><td>
</td><td>
</td></tr>
<tr><td>$\nabla^2 \phi = 0$
$\vec{v} \cdot \hat{n} = \frac{\partial \phi}{\partial n} = \vec{U} \cdot \hat{n} = (U,0,0) \cdot (n_x, n_y, n_z)$
$= U n_x$ on Body
$\vec{v} \to 0$ as $|\vec{x}| \to \infty$
$\phi \to 0$ as $|\vec{x}| \to \infty$</td><td>$\nabla^2 \phi' = 0$
$\vec{v}' \cdot \hat{n}' = \frac{\partial \phi'}{\partial n} = 0$
$\vec{v}' \to (-U,0,0)$ as $|\vec{x}'| \to \infty$
$\phi' \to -Ux'$ as $|\vec{x}'| \to \infty$</td></tr>
<tr><td colspan="2">Galilean transform:
$$\begin{aligned} \vec{v}(x,y,z,t) &= \vec{v}'(x' = x - Ut, y, z, t) + (U,0,0) \\ \phi(x,y,z,t) &= \phi'(x' = x - Ut, y, z, t) + Ux' \Rightarrow \\ -Ux' + \phi(x = x' + Ut, y, z, t) &= \phi'(x', y, z, t) \end{aligned}$$</td></tr>
<tr><td colspan="2">Pressure (no gravity)
$$\begin{aligned} p_\infty = -\tfrac{1}{2}\rho v^2 + C_o = C_o &= -\tfrac{1}{2}\rho v'^2 + C_o' = C_o' - \tfrac{1}{2}\rho U^2 \\ \therefore C_o &= C_o' - \tfrac{1}{2}\rho U^2 \end{aligned}$$</td></tr>
<tr><td>In O: unsteady flow
$p_s = -\rho \frac{\partial \phi}{\partial t} - \frac{1}{2}\rho \underbrace{v^2}_{U^2} + C_o$
$\frac{\partial \phi}{\partial t} = (\underbrace{\frac{\partial}{\partial t}}_{0} + \underbrace{\frac{\partial x'}{\partial t}}_{-U} \frac{\partial}{\partial x'})(\phi' + Ux') = -U^2$
$\therefore p_s = \rho U^2 - \frac{1}{2}\rho U^2 + C_o = \frac{1}{2}\rho U^2 + C_o$
$p_s - p_\infty = \frac{1}{2}\rho U^2$ stagnation pressure</td><td>In O': steady flow
$p_s = -\rho \underbrace{\frac{\partial \phi'}{\partial t}}_{0} - \frac{1}{2}\rho \underbrace{v'^2}_{0} + C_o' = C_o'$
$p_s - p_\infty = \frac{1}{2}\rho U^2$ stagnation pressure</td></tr>
</table>

3.12.2 Forces



Total fluid force for <u>ideal</u> flow (i.e., no shear stresses): $\boxed{\vec{F} = \oiint_B p\hat{n}dS}$

For potential flow, substitute for p from Bernoulli:

$$\vec{F} = \oiint_B -\rho\left(\underbrace{\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2}_{\text{hydrodynamic force}} + \underbrace{gy}_{\text{hydrostatic force}} + c(t)\right)\hat{n}dS$$

For the hydrostatic case $\left(\vec{v} \equiv \phi \equiv 0\right)$:

$$\vec{F}_s = \oiint_B \left(-\rho g y \hat{n}\right) dS \underset{\substack{\uparrow \\ \text{Gauss theorem}}}{=} \underset{\substack{\uparrow \\ \text{outward normal}}}{(-)} \iiint_{\upsilon_B} \nabla\left(-\rho g y\right) d\upsilon = \underbrace{\rho g \forall \hat{j}}_{\text{Archimedes principle}} \quad \text{where } \forall = \iiint_{\upsilon_B} d\upsilon$$

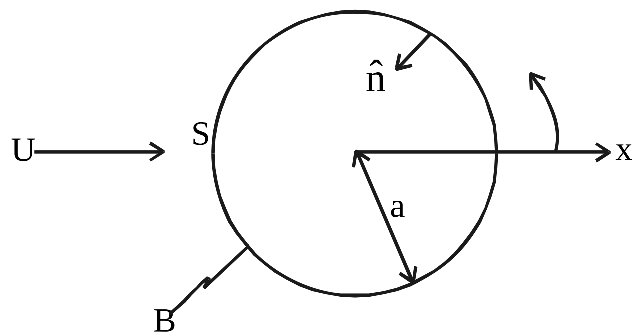
We evaluate **only** the hydrodynamic force:

$$\vec{F}_d = -\rho \iint_B \left(\frac{\partial\phi}{\partial t} + \frac{1}{2}\left|\nabla\phi\right|^2\right)\hat{n}dS$$

For steady motion $\left(\frac{\partial\phi}{\partial t} \equiv 0\right)$:

$$\vec{F}_d = -\frac{1}{2}\rho \iint_B v^2 \hat{n}dS$$

3.12.3 **Example** Hydrodynamic force on 2D cylinder in a steady uniform stream.



$$\vec{F}_d = \int_B \left(\tfrac{-\rho}{2}\right) |\nabla\phi|^2 \hat{n} d\ell = \int_0^{2\pi} \left(\tfrac{-\rho}{2}\right) |\nabla\phi|^2_{r=a} \hat{n} a d\theta$$

$$F_x = \vec{F} \cdot \hat{i} = \tfrac{-\rho a}{2} \int_0^{2\pi} d\theta \, |\nabla\phi|^2_{r=a} \underbrace{\hat{n} \cdot \hat{i}}_{-\cos\theta}$$

$$= \tfrac{\rho a}{2} \int_0^{2\pi} |\nabla\phi|^2_{r=a} \cos\theta d\theta$$

Velocity potential for flow past a 2D cylinder:

$$\phi = Ur\cos\theta \left(1 + \frac{a^2}{r^2}\right)$$

Velocity vector on the 2D cylinder surface:

$$\nabla\phi|_{r=a} = \left(v_r|_{r=a}, v_\theta|_{r=a}\right) = \left(\underbrace{\left.\frac{\partial\phi}{\partial r}\right|_{r=a}}_{0}, \underbrace{\left.\frac{1}{r}\frac{\partial\phi}{\partial\theta}\right|_{r=a}}_{-2U\sin\theta}\right)$$

Square of the velocity vector on the 2D cylinder surface:

$$\left.|\nabla\phi|^2\right|_{r=a} = 4U^2\sin^2\theta$$

Finally, the **hydrodynamic force** on the 2D cylinder is given by

$$F_x = \frac{\rho a}{2}\int_0^{2\pi} d\theta\left(4U^2\sin^2\theta\cos\theta\right) = \underbrace{\left(\frac{1}{2}\rho U^2\right)}_{p_s - p_\infty}\underbrace{(2a)}_{\substack{\text{diameter}\\ \text{or}\\ \text{projection}}}\underbrace{2\int_0^{2\pi} d\theta\underbrace{\sin^2\theta}_{\text{even}}\underbrace{\cos\theta}_{\substack{\text{odd}\\ \text{w.r.t}\frac{\pi}{2},\frac{3\pi}{2}}}}_{\equiv 0} = 0$$

Therefore, $F_x = 0 \Rightarrow$ no horizontal force (symmetry fore-aft of the streamlines). Similarly,

$$F_y = \left(\frac{1}{2}\rho U^2\right)(2a)2\int_0^{2\pi} d\theta\sin^2\theta\sin\theta = 0$$

In fact, *in general* we find that $\vec{F} \equiv 0$, on *any* 2D or 3D body.

D'Alembert's "paradox":

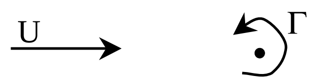
No hydrodynamic force* acts on a body moving with steady translational (no circulation) velocity in an infinite, inviscid, irrotational fluid.

* The moment as measured in a local frame is not necessarily zero.

3.13 Lift due to Circulation

3.13.1 **Example** Hydrodynamic force on a vortex in a uniform stream.

$$\phi = Ux + \frac{\Gamma}{2\pi}\theta = Ur\cos\theta + \frac{\Gamma}{2\pi}\theta$$

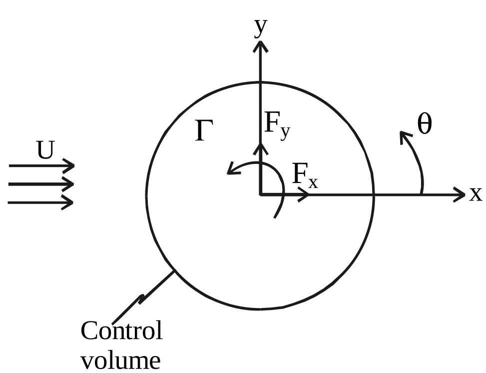


Consider a control surface in the form of a circle of radius r centered at the point vortex. Then according to Newton's law:

$$\begin{aligned} \Sigma\vec{F} &= \frac{d}{dt}\vec{\mathcal{L}}_{CV} \xrightarrow{\text{steady flow}} \\ (\vec{F}_V + \vec{F}_{CS}) + \vec{M}_{NET} &= 0 \Leftrightarrow \vec{F} \equiv -\vec{F}_V = \vec{F}_{CS} + \vec{M}_{NET} \end{aligned}$$

Where,

$\vec{F}$	= Hydrodynamic force exerted on the vortex from the fluid.
$\vec{F}_V = -\vec{F}$	= Hydrodynamic force exerted on the fluid in the control volume from the vortex.
$\vec{F}_{CS}$	= Surface force (i.e., pressure) on the fluid control surface.
$\vec{M}_{NET}$	= Net linear momentum flux in the control volume through the control surface.
$\frac{d}{dt}\vec{\mathcal{L}}_{CV}$	= Rate of change of the total linear momentum in the control volume.



The hydrodynamic force on the vortex is $\vec{F} = \vec{F}_{CS} + \vec{M}_{IN}$

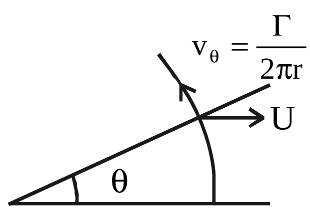
a. Net linear momentum flux in the control volume through the control surfaces, $\vec{M}_{NET}$. Recall that the control surface has the form of a circle of radius r centered at the point vortex.

a.1 The velocity components on the control surface are

$$u = U - \frac{\Gamma}{2\pi r}\sin\theta$$
$$v = \frac{\Gamma}{2\pi r}\cos\theta$$

The radial velocity on the control surface is therefore, given by

$$u_r = U\frac{\partial x}{\partial r} = U\cos\theta = \vec{V}\cdot\hat{n}$$



a.2 The net horizontal and vertical momentum fluxes through the control surface are given by

$$(M_{NET})_x = -\rho\int_0^{2\pi} d\theta r u v_r = -\rho\int_0^{2\pi} d\theta r\left(U - \frac{\Gamma}{2\pi r}\sin\theta\right)U\cos\theta = 0$$
$$(MNET)_y = -\rho\int_0^{2\pi} d\theta r v v_r = -\rho\int_0^{2\pi} d\theta r\left(\frac{\Gamma}{2\pi r}\cos\theta\right)U\cos\theta$$
$$= -\frac{\rho U\Gamma}{2\pi}\int_0^{2\pi}\cos^2\theta d\theta = -\frac{\rho U\Gamma}{2}$$

b. Pressure force on the control surface, $\vec{F}_{CS}$.

b.1 From Bernoulli, the pressure on the control surface is

$$p = -\frac{1}{2}\rho\,|\vec{v}|^2 + C$$

b.2 The velocity $|\vec{v}|^2$ on the control surface is given by

$$\begin{aligned}|\vec{v}|^2 =& u^2 + v^2 = \left(U - \frac{\Gamma}{2\pi r}\sin\theta\right)^2 + \left(\frac{\Gamma}{2\pi r}\cos\theta\right)^2 \\ =& U^2 - \frac{\Gamma}{\pi r}U\sin\theta + \left(\frac{\Gamma}{2\pi r}\right)^2\end{aligned}$$

b.3 Integrate the pressure along the control surface to obtain $\vec{F}_{CS}$

$$\begin{aligned}(F_{CS})_x &= \int_0^{2\pi} d\theta r p(-\cos\theta) = 0 \\ (F_{CS})_y &= \int_0^{2\pi} d\theta r p(-\sin\theta) = \left(-\tfrac{\rho}{2}\right)\left(-\tfrac{\Gamma U}{\pi r}\right)(-r)\underbrace{\int_0^{2\pi} d\theta \sin^2\theta}_{\pi} = -\tfrac{1}{2}\rho U\Gamma\end{aligned}$$

c. Finally, the force on the vortex $\vec{F}$ is given by

$$\begin{aligned}F_x &= (F_{CS})_x + (M_x)_{IN} = 0 \\ F_y &= (F_{CS})_y + (M_y)_{IN} = -\rho U\Gamma\end{aligned}$$

i.e., the fluid exerts a downward force $F = -\rho U\Gamma$ on the vortex.

Kutta-Joukowski Law

$$\begin{aligned}2D: &\quad F = -\rho U\Gamma \\ 3D: &\quad \vec{F} = \rho\vec{U}\times\vec{\Gamma}\end{aligned}$$

Generalized Kutta-Joukowski Law:

$$\vec{F} = \rho\vec{U}\times\left(\sum_{i=1}^{n}\vec{\Gamma}_i\right)$$

where $\vec{F}$ is the total force on a system of n vortices in a free stream with speed $\vec{U}$.