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2.161 Signal Processing: Continuous and Discrete
Fall 2008

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MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.161 *Signal Processing - Continuous and Discrete*
Fall Term 2008

Lecture 16[1]

Reading:

- Proakis & Manolakis, Sec. 10.2
- Oppenheim, Schafer & Buck, Chap. 7.
- Cartinhour, Chap. 9.

1 FIR Low-Pass Filter Design by Windowing

In Lecture 15 we examined the creation of a causal FIR filter based upon an ideal low-pass filter with cut-off frequency ω_c, and found that the impulse response was

$$h(n) = \frac{\omega_c}{\pi}\left(\frac{\sin(\omega_c n}{\Omega_c n}\right) \qquad -\infty < n < \infty.$$

The resulting filter is therefore both infinite in extent and non-causal.

To create a finite length filter we truncated the impulse response by multiplying $\{h(n)\}$ with an even rectangular window function $\{r(n)\}$ of length $M+1$, where

$$r(n) = \begin{cases} 1 & |n| \le M/2 \\ 0 & \text{otherwise.} \end{cases}$$

The result was to create a modified filter $\{h'_n\}$ with a real frequency response function $H'(\mathrm{e}^{\mathrm{j}\omega})$ from the convolution

$$H'(\mathrm{e}^{\mathrm{j}\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\mathrm{e}^{\mathrm{j}\nu})R(\mathrm{e}^{\mathrm{j}(\omega-\nu)})\,\mathrm{d}\nu$$

where

$$R(\mathrm{e}^{\mathrm{j}\omega}) = \frac{\sin((M+1)\omega/2)}{\sin(\omega/2)}$$

The truncation generates a Gibb's phenomenon associated with the band edges of $H'(\mathrm{e}^{\mathrm{j}\omega})$ where, as demonstrated in the figure below:

(a) Both the pass-band and the stop-band exhibit significant ripple, and the maxima of the ripple is relatively independent of the chosen filter length $M+1$.

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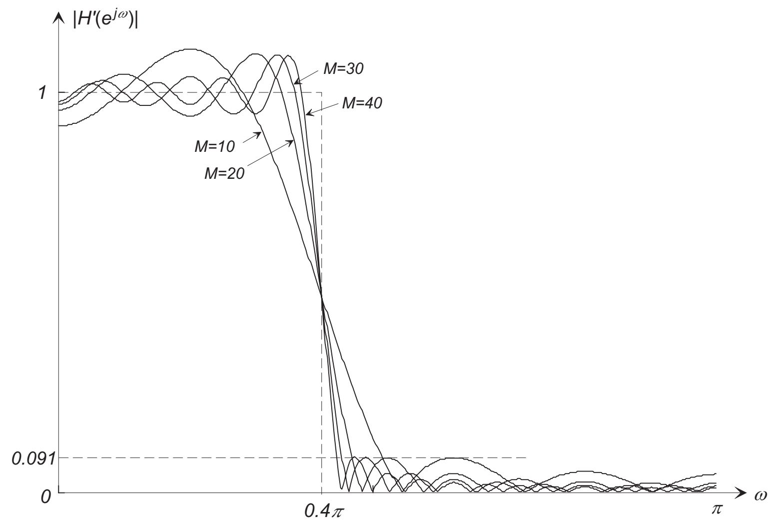
(b) The amplitude of the first side-lobe in the stop-band ia approximately 0.091, corresponding to an attenuation of 21 dB, at that frequency.

(c) The width of the transition region decreases with $M+1$, the filter length.



A causal filter was then formed by applying a right-shift of $M/2$ to the impulse response to form $\{\hat{h}_n\}$ where

$$\hat{h}(n) = h'(n - M/2) \qquad 0 \le n \le M+1.$$

The shift was seen to have no effect on $\left|H(\mathrm{e}^{\mathrm{j}\,\omega})\right|$, but created a linear phase taper (lag).

The *windowing method* of FIR seeks to improve the filter characteristic by selecting an alternate length $M+1$ window function $\{w(n)\}$ with improved spectral characteristics $W(\mathrm{e}^{\mathrm{j}\,\omega})$, which when convolved with the ideal low-pass filter function $\left|H(\mathrm{e}^{\mathrm{j}\,\omega})\right|$ will produce a "better" filter.

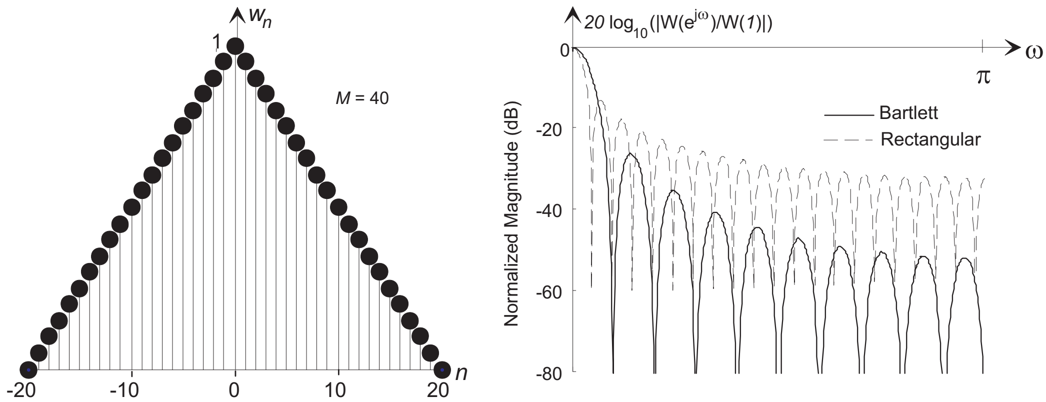
There are many window functions available. We first look at three common fixed parameter windows:

The Bartlett Window: The length $M+1$ Bartlett window is a even triangular window

$$w(n) = \begin{cases} 1 + 2n/M & -M/2 \le n \le 0 \\ 1 - 2n/M & 0 \le n \le M/2 \\ 0 & \text{otherwise,} \end{cases}$$

as shown for $M+1 = 40$ in the figure below. Also plotted is the spectrum $\left|W(\mathrm{e}^{\mathrm{j}\,\omega}\right|$, and for comparison the spectrum of the same length rectangular window $\left|R(\mathrm{e}^{\mathrm{j}\,\omega}\right|$.

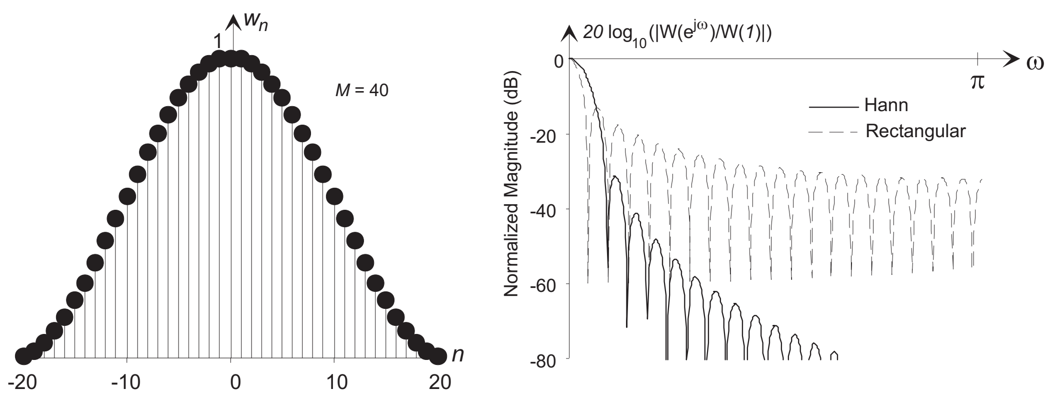
It can be seen that the main lobe of the Bartlett spectrum is wider than that of the rectangular window, but that the side-lobes decrease in amplitude much faster at higher frequencies. The Bartlett window produces a monotonically decreasing frequency response magnitude, as is shown below.



The Hann (or "Hanning") Window: The Hann window is a raised cosine window

$$w(n) = \begin{cases} 0.5 + 0.5\cos\left(\frac{2\pi}{M}n\right) & -M/2 \le n \le M/2 \\ 0 & \text{otherwise.} \end{cases}$$

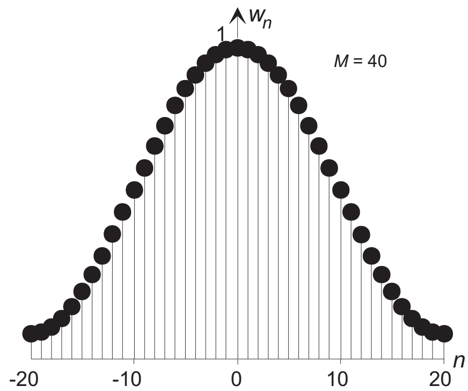
The Hann window, along with its spectrum, is shown for $M + 1 = 40$ below. As with the Bartlett example above, the spectrum of the rectangular window is given for comparison. Again it can be seen that the Hann window has a broader main lobe, but with much reduced side-lobes (even compared to the Bartlett window) away from the main peak.

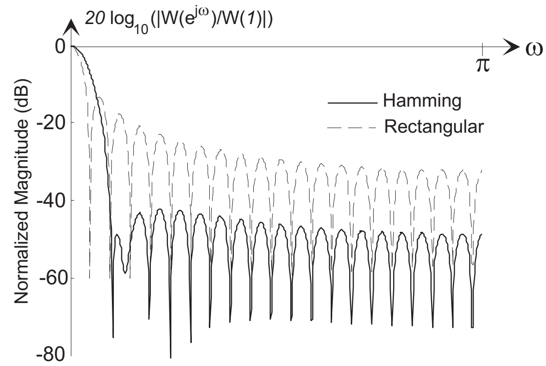


The Hamming Window: The Hamming window is another raised cosine window, but this time on a pedestal.

$$w(n) = \begin{cases} 0.54 + 0.46\cos\left(\frac{2\pi}{M}n\right) & -M/2 \le n \le M/2 \\ 0 & \text{otherwise.} \end{cases}$$

so that at the extremities ($n = \pm M/2$), the value $w_{\pm M/2} = 0.08$. From the figure below, it can be seen that the Hamming widow has smaller side-lobes close to the main lobe, but that the side-lobes distant from the main peak have a higher amplitude.





Filter Design Procedure Using a Fixed Window:
The only design parameters available when using a fixed window are (1) the low-pass cut-off frequency ω_c, (2) the choice of window type, and (3) the filter length $M+1$. Once these choices are made, the procedure is as follows

(a) Form the samples of the ideal low-pass filter of length $M+1$.

$$h(n) = \frac{\omega_c}{\pi}\left(\frac{\sin(\omega_c n)}{\omega_c n}\right) \qquad \text{for } -M/2 \le n \le M/2$$

(b) Form the length $M+1$ window $\{w_n\}$ of the chosen type.

(c) Form the impulse response $\{h'_n\}$ where $h'_n = h_n w_n$.

(d) Shift all samples to the right by $M/2$ samples.

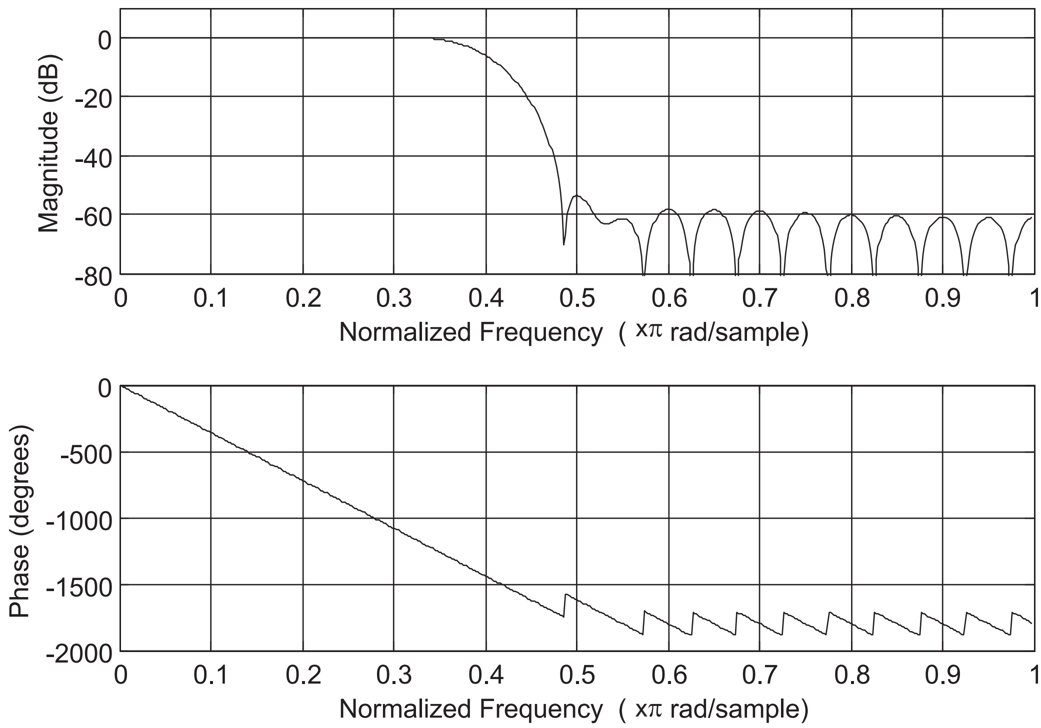
■ Example 1

Write some MATLAB code to design a length 41 low-pass FIR filter with cut-off frequency $\omega_c = 0.4\pi$ using a Hamming window. Plot the magnitude and pahase of the resulting filter.

Solution: The following MATLAB code was used:

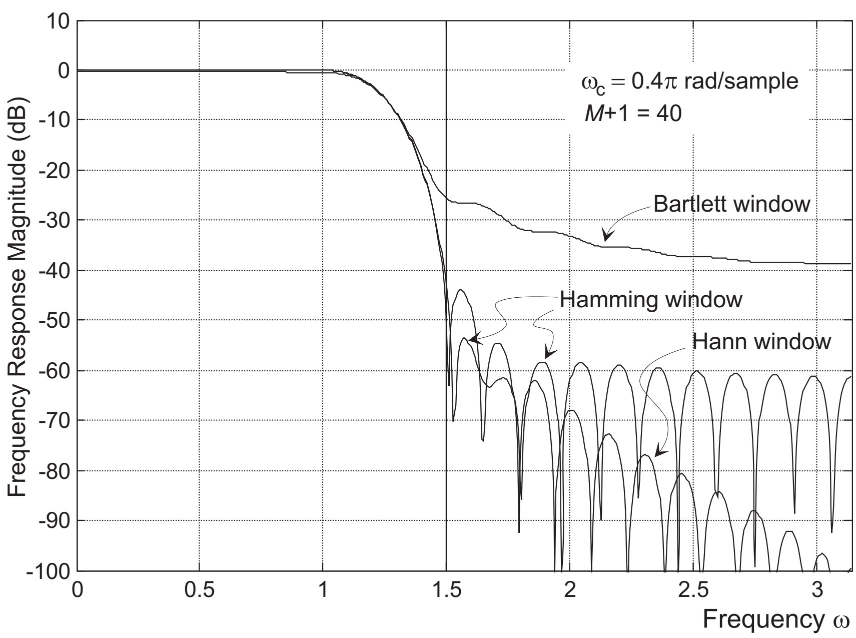
```
n=-20:20;
wc=0.4*pi;
h = (wc/pi)*sinc(wc/pi*n);
hprime = h.*hamming(41)';
% All done - no need to shift - just interpret hprime as the shifted
% impulse response.
% Plot the frequency response:
freqz(hprime,1);
```

which generated the following frequency response plots:



Note that the linear phase characteristic has jump discontinuities of π (or $180°$) when $H'(e^{j\omega})$ changes sign.

The following figure shows a comparison of length 41 filters designed with the Bartlett, Hann, and Hamming windows.

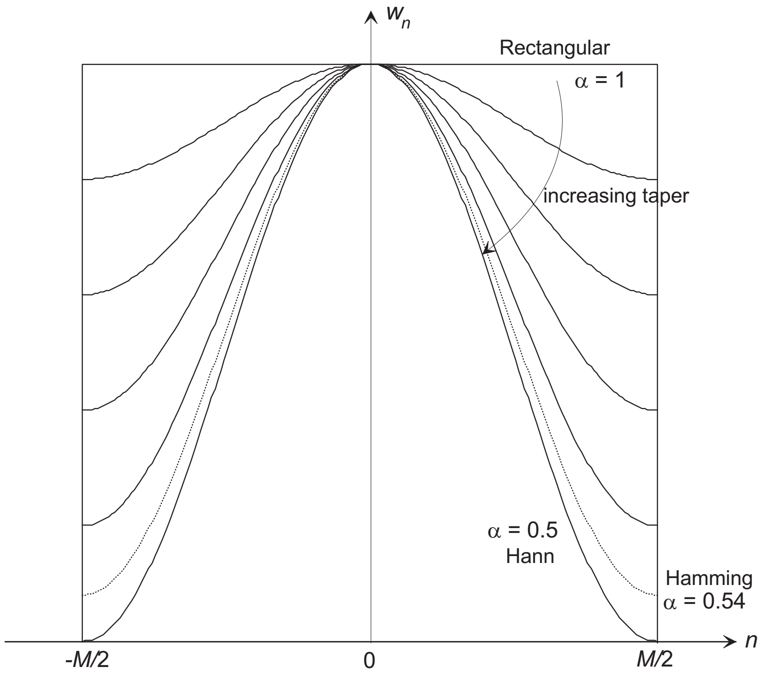


Notice that while the Bartlett window generates a filter with less attenuation in the stop-band, it has no ripple in the stop-band (no sign changes in $H'(\mathrm{e}^{j\omega})$) and therefore no jump discontinuities in its linear phase characteristic.

General Comments on Window Taper Consider the family of window functions that are raised cosine functions on a pedestal, characterized by

$$w_\alpha(n) = \begin{cases} \alpha + (1-\alpha)\cos\left(\frac{2\pi}{M}n\right) & -M/2 \le n \le M/2 \\ 0 & \text{otherwise.} \end{cases}$$

where the parameter α, for $1 \ge \alpha \ge 0.5$, defines the degree of taper. When $\alpha = 1$ we have the rectangular window with zero taper, when $\alpha = 0.5$ we have the Hann window (maximum taper) , and the Hamming window corresponds to $\alpha = 0.54$.

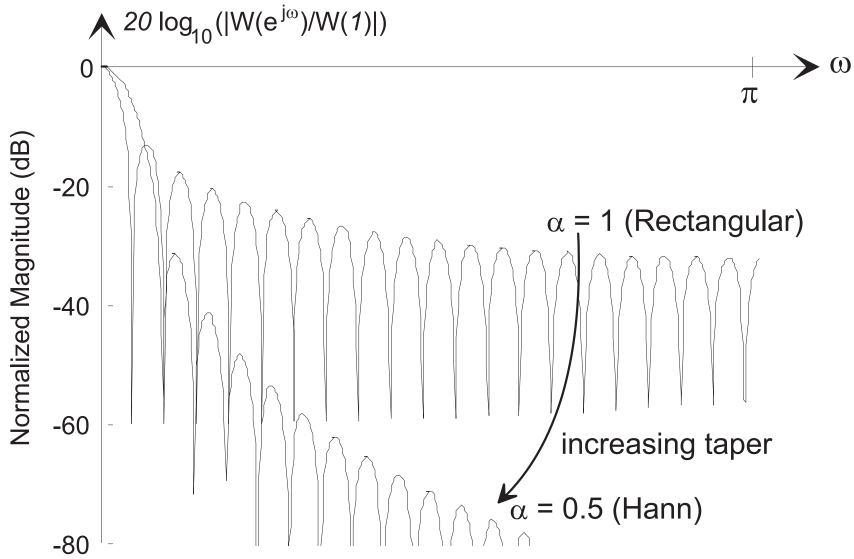


These window functions may written as a linear combination of the rectangular window $w_{\mathrm{rect}}(n)$, and the Hann window $w_{\mathrm{Hann}}(n)$:

$$w_\alpha(n) = 2(1-\alpha)w_{\mathrm{Hann}}(n) + 2(\alpha - 0.5)w_{\mathrm{rect}}(n)$$

The spectra of these windows $W_\alpha(\mathrm{e}^{j\omega})$ will therefore be a similar combination of the spectra $W_{\mathrm{rect}}(\mathrm{e}^{j\omega})$ of the rectangular window, and $W_{\mathrm{Hann}}(\mathrm{e}^{j\omega})$ of the Hann window.

$$W_\alpha(\mathrm{e}^{j\omega}) = 2(1-\alpha)W_{\mathrm{Hann}}(\mathrm{e}^{j\omega}) + 2(\alpha - 0.5)W_{\mathrm{rect}}(\mathrm{e}^{j\omega})$$



Although we have only discussed raised cosine windows here, in general the degree of taper affects the convolution kernel as follows:

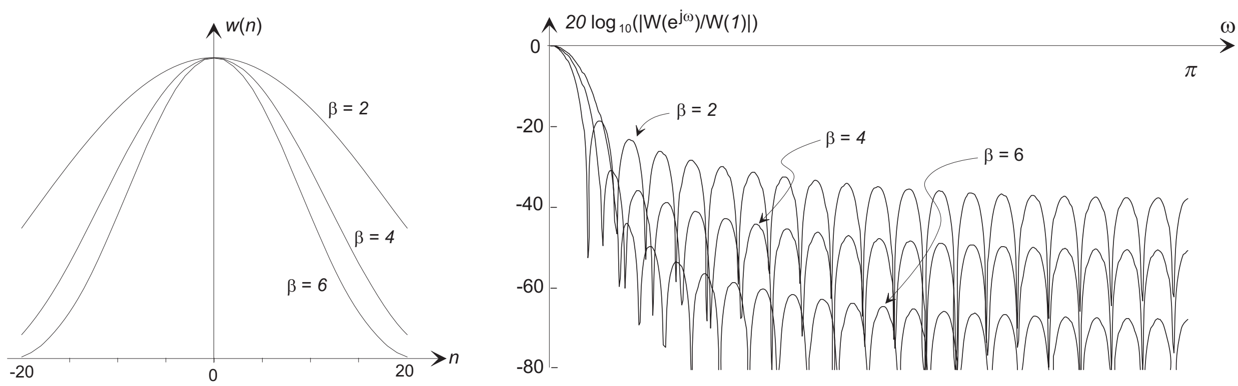
- As the taper increases the width of the main lobe increases, causing the transition band-width in the filter to increase.
- As the taper increases the amplitude of the side-lobes decreases more rapidly away from the main lobe, with the result that the filter stop-band attenuation is significantly increased at high frequencies.

The Kaiser Window: The Kaiser window, defined as

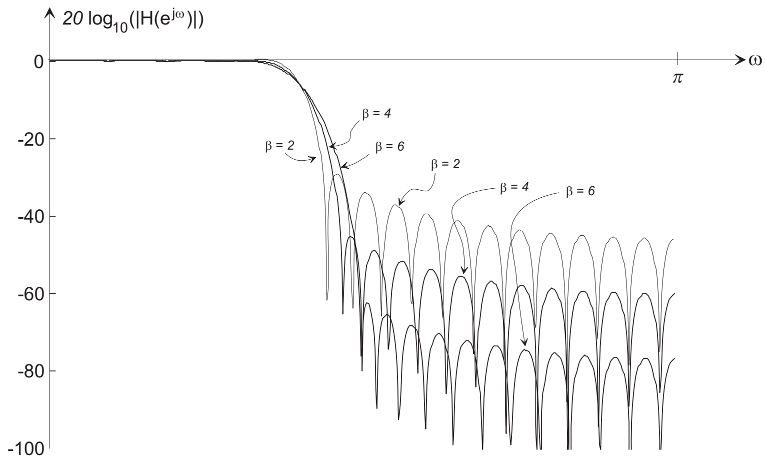
$$w_K(n,\beta) = \begin{cases} \frac{I_0\left(\beta\sqrt{1-\left(\frac{n}{M/2}\right)^2}\right)}{I_0(\beta)}, & -\frac{M}{2} \le n \le \frac{M}{2} \\ 0, & \text{otherwise} \end{cases}$$

where $I_0()$ is the zero-order modified Bessel function of the first kind, and the parameter β provides a convenient control over the window taper (and the resultant trade-off between lower side-lobe amplitudes and the width of the main lobe). (Note: Some authors define the window in terms of a parameter $\alpha = 2\beta/M$.)

Kaiser windows for $\beta = 2, 4, 6$ are shown below:



The effect of the parameter β on the window taper, and the compromise between the width of the main lobe and sidelobe amplitude can be easily seen. These three window functions were used to design low-pass FIR filters with $\omega_c = 0.4\pi$. the frequency response magnitudes are shown below

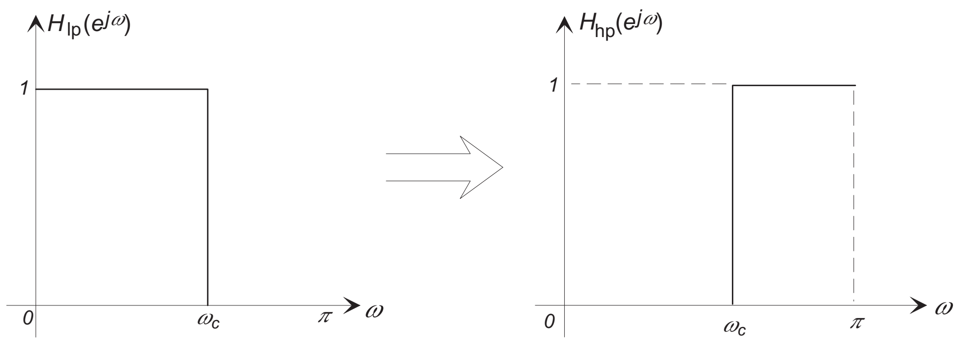


The compromise between stop-band attenuation and transition steepness can be clearly seen. The Kaiser window is very commonly used in FIR filters.

2 Window FIR Filters or Other Filter Types

High-Pass Filter: Given an ideal low-pass filter $H_{lp}(\mathrm{e}^{\mathrm{j}\omega})$, a high-pass filter $H_{hp}(\mathrm{e}^{\mathrm{j}\omega})$ may be created:

$$H_{hp}(\mathrm{e}^{\mathrm{j}\omega}) = 1 - H_{lp}(\mathrm{e}^{\mathrm{j}\omega})$$



Then the impulse response is

$$\begin{aligned} \{h_{hp}(n)\} &= \mathrm{IDFT}\{1\} - \mathrm{IDFT}\left\{H_{lp}(\mathrm{e}^{\mathrm{j}\omega})\right\} \\ &= \delta(n) - \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n} \end{aligned}$$

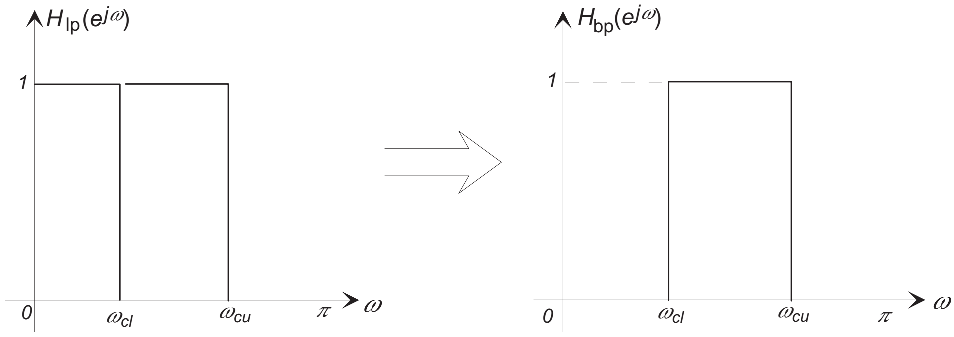
After windowing to a length $M+1$

$$h(n) = w(n)\left(\delta(n) - \frac{\omega_c}{\pi}\frac{\sin(\omega_c n)}{\omega_c n}\right), \qquad |n| \le M/2.$$

The impulse response is then shifted to the right by $M/2$ samples to make it causal as before.

Band-Pass Filter: A band-pass filter $H_{bp}(\mathrm{e}^{\mathrm{j}\omega})$ may be designed from a pair of low-pass filters $H_{lpu}(\mathrm{e}^{\mathrm{j}\omega})$ and $H_{lpl}(\mathrm{e}^{\mathrm{j}\omega})$ with cut-off frequencies ω_{cu} and ω_{cl} respectively,

$$H_{bp}(\mathrm{e}^{\mathrm{j}\omega}) = H_{lpu}(\mathrm{e}^{\mathrm{j}\omega}) - H_{lpl}(\mathrm{e}^{\mathrm{j}\omega}).$$

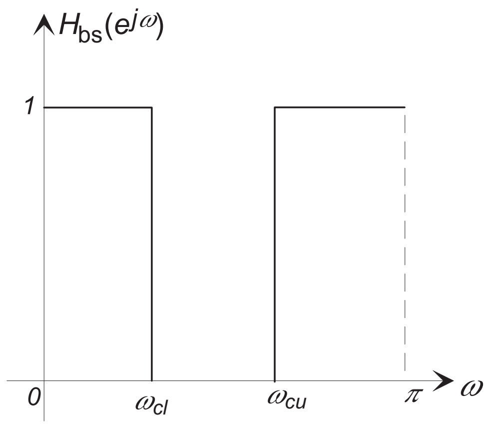


Then

$$h_{bp}(n) = w(n)\left(\frac{\omega_{cu}}{\pi}\frac{\sin(\omega_{cu}n)}{\omega_{cu}n} - \frac{\omega_{cl}}{\pi}\frac{\sin(\omega_{cl}n)}{\omega_{cl}n}\right), \qquad |n| \leq M/2.$$

Band-stop Filter: A band-stop filter $H_{bs}(\mathrm{e}^{\mathrm{j}\omega})$ may be designed from a low-pass filters $H_{lp}(\mathrm{e}^{\mathrm{j}\omega})$ and a high-pass filter $H_{hp}(\mathrm{e}^{\mathrm{j}\omega})$ with cut-off frequencies ω_{cl} and ω_{cu} respectively,

$$H_{bs}(\mathrm{e}^{\mathrm{j}\omega}) = H_{lp}(\mathrm{e}^{\mathrm{j}\omega}) + H_{hp}(\mathrm{e}^{\mathrm{j}\omega}).$$



Then

$$h_{bs}(n) = w(n)\left(\frac{\omega_{cu}}{\pi}\frac{\sin(\omega_{cu}n)}{\omega_{cu}n} + \delta(n) - \frac{\omega_{cl}}{\pi}\frac{\sin(\omega_{cl}n)}{\omega_{cl}n}\right), \qquad |n| \leq M/2.$$

We show below that a linear phase high-pass or band-stop filter must have a length $M+1$ that is odd.

3 The Zeros of a Linear Phase FIR Filter

Consider the transfer function of a FIR system with an *even-symmetric* impulse response of length $M+1$

$$H(z) = \sum_{k=0}^{M} h_k z^{-k}$$

The order of the polynomial is M. Also

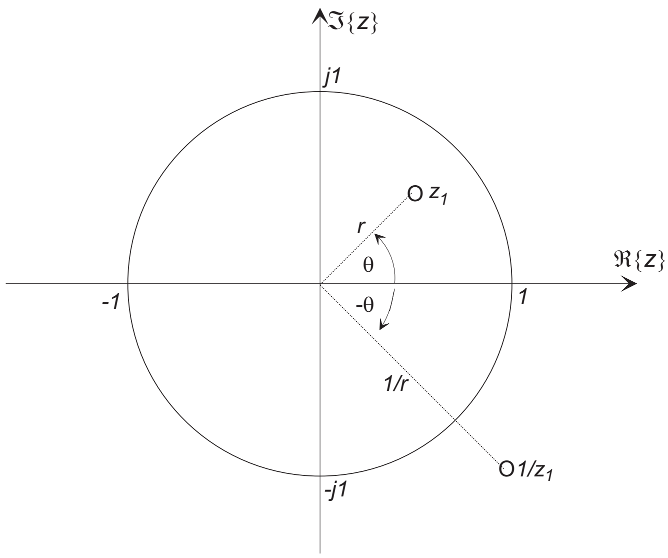
$$H(z^{-1}) = \sum_{k=0}^{M} h_k z^k = z^M \sum_{k=o}^{M} h_k z^{-(M-k)} = z^M \sum_{n=o}^{M} h_{M-n} z^{-n}$$

where $n = M - k$. Because $\{h_k\}$ is even-symmetric, $h_k = h_{M-k}$, and the polynomials in $H(z)$ and $H(z^{-1})$ are identical

$$\boxed{H(z^{-1}) = z^{(M)} H(z).}$$

This means that if z_1 is a zero of $H(z)$, that is $H(z_1) = 0$, then also $H(1/z_1) = 0$, and therefore $1/z_1$ is also a zero of $H(z)$.

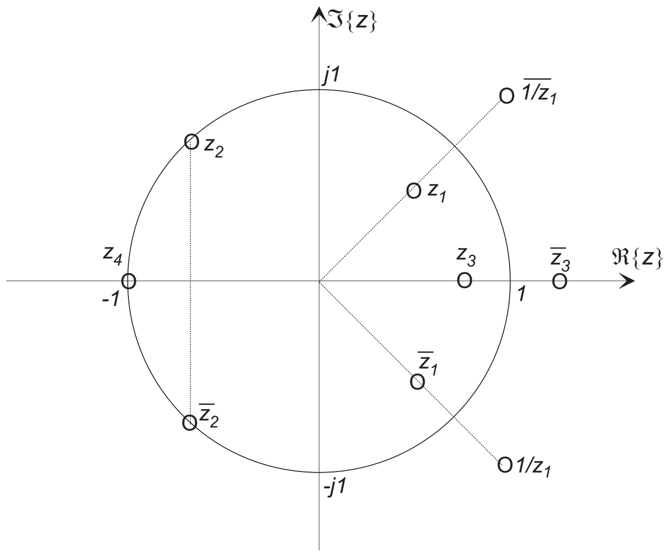
If $z_1 = r\,\mathrm{e}^{\mathrm{j}\,\theta}$, then $1/z_1 = (1/r)\,\mathrm{e}^{-\mathrm{j}\,\theta}$ and the reciprocal zeros may be drawn on the z-plane



In addition, zeros are either real or appear in complex conjugate pairs, with the result

- A general complex zero will be a member of a group of four zeros that are a "quad" of reciprocal conjugates.
- A pair complex zeros on the unit circle are their own reciprocals, and so will exist only as a pair.
- A general real zero will be a member of a conjugate pair.
- A zero at $z = \pm 1$ will satisfy its own reciprocal, and therefore may exist on its own.

The figure below shows a quad of zeros associated with a complex zero z_1, a conjugate pair of zeros on the unit circle associated with z_2, a reciprocal pair associated with the real zero z_3, and a single zero z_4 at $z = -1$.



In addition

$$H(z) = \sum_{k=0}^{M} h_k z^{-k} = \frac{1}{z^M} \sum_{k=0}^{M} h_k z^{M-k} = \frac{1}{z^M} \sum_{n=0}^{M} h_{M-n} z^n$$

where $n = M - k$. But with even symmetry $h(M-n) = h(n)$, and since $(-1)^k = (-1)^{-k}$, at $z = -1$

$$H(-1) = (-1)^{-M} H(-1).$$

If M is odd, $H(-1) = -H(-1)$, thus forcing $H(-1) = 0$, therefore $H(z)$ has a zero at $z = -1$ if the filter length $M + 1$ is even.

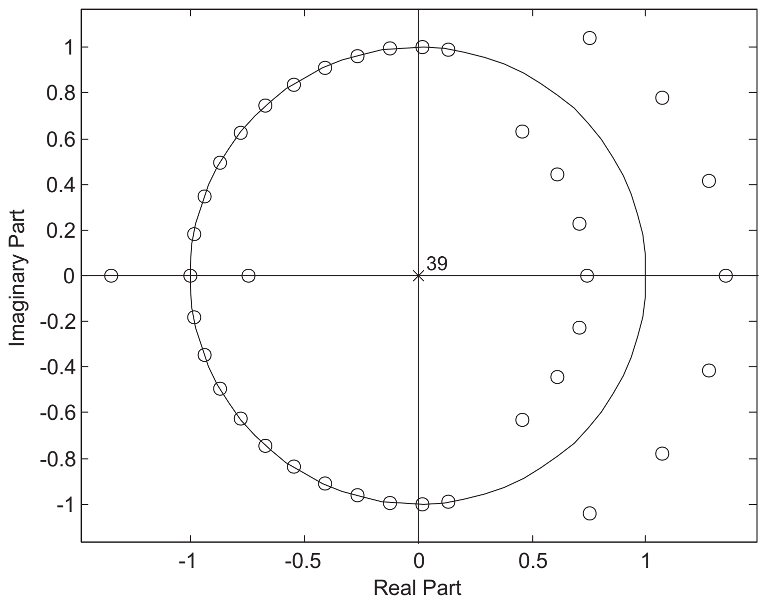
Any filter with a finite response magnitude at $\omega = \pi$ cannot have a zero at $z = -1$. For an even-symmetric FIR filter this requires that the filter length $M + 1$ be odd (or equivalently that the number of zeros be even). Linear phase FIR high-pass and band-stop filters *must* have an odd filter length.

■ Example 2

Draw the pole-zero plot for a length 40 low-pass linear-phase FIR filter with $\omega_c = 0.4\imath$ using a Kaiser window with $\beta = 3$.

Solution: The plot below was generated with the MATLAB commands:

```
>> b=fir1(39,0.4,kaiser(40,3));
>> zplane(b,1)
```



The complex reciprocal conjugate quads in the pass-band, conjugate pairs on the unit circle in the stop-band, and real axis reciprocals can be clearly seen.

Notice that because $M + 1 = 40$ is even, there is a zero at $z = -1$, and that this filter would not be satisfactory for transformation to a high-pass or band-stop filter.