2.161 Signal Processing: Continuous and Discrete
Fall 2008

Massachusetts Institute of Technology

Department of Mechanical Engineering

2.161 *Signal Processing - Continuous and Discrete*

Fall Term 2008

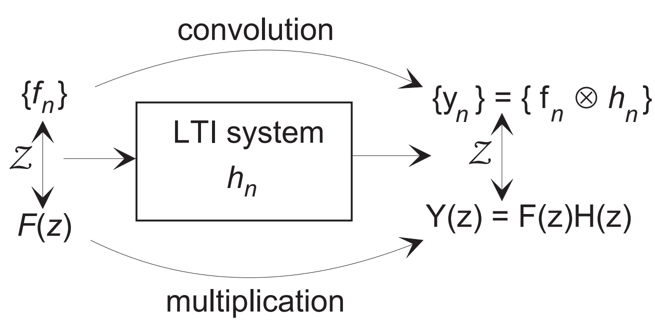
Lecture 14[1]

Reading:

- Proakis & Manolakis, Chapter 3 (The z-transform)
- Oppenheim, Schafer & Buck, Chapter 3 (The z-transform)

1 The Discrete-Time Transfer Function

Consider the discrete-time LTI system, characterized by its pulse response $\{h_n\}$:



We saw in Lec. 13 that the output to an input sequence $\{f_n\}$ is given by the convolution sum:

$$y_n = f_n \otimes h_n = \sum_{k=-\infty}^{\infty} f_k h_{n-k} = \sum_{k=-\infty}^{\infty} h_k f_{n-k},$$

where $\{h_n\}$ is the pulse response. Using the convolution property of the z-transform we have at the output

$$Y(z) = F(z)H(z)$$

where $F(z) = \mathcal{Z}\{f_n\}$, and $H(z) = \mathcal{Z}\{h_n\}$. Then

$$\boxed{H(z) = \frac{Y(z)}{F(z)}}$$

is the *discrete-time transfer function*, and serves the same role in the design and analysis of discrete-time systems as the Laplace based transfer function $H(s)$ does in continuous systems.

In general, for LTI systems the transfer function will be a rational function of z, and may be written in terms of z or z^{-1}, for example

$$H(z) = \frac{N(s)}{D(s)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

where the $b_i, \quad i = 0, \ldots, m,\ a_i, \quad i = 0, \ldots, n$ are constant coefficients.

2 The Transfer Function and the Difference Equation

As defined above, let

$$H(z) = \frac{Y(z)}{F(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}}{a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}}$$

and rewrite as

$$\left(a_0 + a_1 z^{-1} + a_2 z^{-2} + \ldots + a_N z^{-N}\right) Y(z) = \left(b_0 + b_1 z^{-1} + b_2 z^{-2} + \ldots + b_M z^{-M}\right) F(z)$$

If we apply the z-transform time-shift property

$$Z\left\{f_{n-k}\right\} = z^{-k} F(z)$$

term-by-term on both sides of the equation, (effectively taking the inverse z-transform)

$$a_0 y_n + a_1 y_{n-1} + a_2 y_{n-2} + \ldots + a_N y_{n-N} = b_0 f_n + b_1 f_{n-1} + b_2 f_{n-2} + \ldots + b_M f_{n-M}$$

and solve for y_n

$$\begin{aligned} y_n &= -\frac{1}{a_0}\left(a_1 y_{n-1} + a_2 y_{n-2} + \ldots + a_N y_{n-N}\right) + \frac{1}{a_0}\left(b_0 f_n + b_1 f_{n-1} + b_2 f_{n-2} + \ldots + b_M f_{n-M}\right) \\ &= \sum_{i=1}^{N}\left(\frac{-a_i}{a_0}\right) y_{n-i} + \sum_{i=0}^{M}\left(\frac{b_i}{a_0}\right) f_{n-i} \end{aligned}$$

which is in the form of a recursive linear difference equation as discussed in Lecture 13.

The transfer function $H(z)$ directly defines the computational difference equation used to implement a LTI system.

■ Example 1

Find the difference equation to implement a causal LTI system with a transfer function

$$H(z) = \frac{(1 - 2z^{-1})(1 - 4z^{-1})}{z(1 - \frac{1}{2}z^{-1})}$$

Solution:

$$H(z) = \frac{z^{-1} - 6z^{-2} + 8z^{-3}}{1 - \frac{1}{2}z^{-1}}$$

from which

$$y_n - \frac{1}{2}y_{n-1} = f_{n-1} - 6f_{n-2} + 8f_{n-3},$$

or

$$\boxed{y_n = \frac{1}{2}y_{n-1} + (f_{n-1} - 6f_{n-2} + 8f_{n-3}).}$$

The reverse holds as well: if we are given the difference equation, we can define the system transfer function.

■ Example 2

Find the transfer function (expressed in powers of z) for the difference equation

$$y_n = 0.25y_{n-2} + 3f_n - 3f_{n-1}$$

and plot the system poles and zeros on the z-plane.

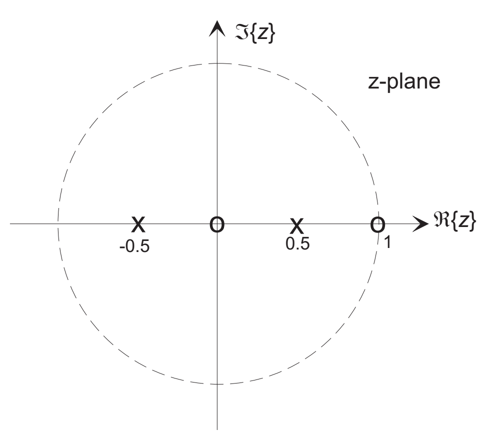
Solution: Taking the z-transform of both sides

$$Y(z) = 0.25z^{-2}Y(z) + 3F(z) - 3z^{-1}F(z)$$

and reorganizing

$$H(z) = \frac{Y(z)}{F(z)} = \frac{3(1 - z^{-1})}{1 - 0.25z^{-2}} = \frac{3z(z-1)}{z^2 - 0.25}$$

which has zeros at $z = 0, 1$ and poles at $z = -0.5, 0.5$:



---

3 Introduction to z-plane Stability Criteria

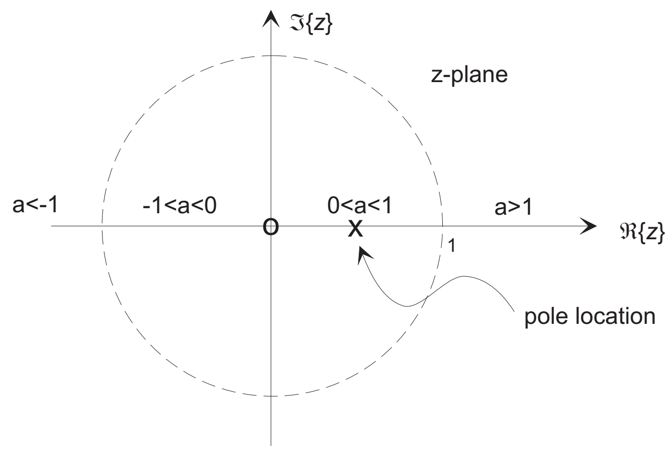
The stability of continuous time systems is governed by pole locations - for a system to be BIBO stable all poles must lie in the l.h. s-plane. Here we do a preliminary investigation of stability of discrete-time systems, based on z-plane pole locations of $H(z)$.

Consider the pulse response h_n of the causal system with

$$H(z) = \frac{z}{z-a} = \frac{1}{1-az^{-1}}$$

with a single real pole at $z = a$ and with a difference equation

$$y_n = ay_{n-1} + f_n.$$



Clearly the pulse response is

$$h_n = \begin{cases} 1 & n = 0 \\ a^n & n \geq 1 \end{cases}$$

The nature of the pulse response will depend on the pole location:

$0 < a < 1$**:** In this case $h_n = a^n$ will be a decreasing function of n and $\lim_{n\to\infty} h_n = 0$ and the system is **stable**.

$a = 1$**:** The difference equation is $y_n = y_{n-1} + f_n$ (the system is a summer and the impulse response is $h_n = 1$, (non-decaying). The system is **marginally stable**.

$a > 1$**:** In this case $h_n = a^n$ will be a increasing function of n and $\lim_{n\to\infty} h_n = \infty$ and the system is **unstable**.

$-1 < a < 0$**:** In this case $h_n = a^n$ will be a oscillating but decreasing function of n and $\lim_{n\to\infty} h_n = 0$ and the system is **stable**.

$a = -1$**:** The difference equation is $y_n = -y_{n-1} + f_n$ and the impulse response is $h_n = (-1)^n$, that is a pure oscillator. The system is **marginally stable**.

$a < -1$**:** In this case $h_n = a^n$ will be a oscillating but increasing function of n and $\lim_{n\to\infty} |h_n| = \infty$ and the system is **unstable**.

This simple demonstration shows that this system is stable only for the pole position $-1 < a < 1$. In general for a system

$$H(z) = K \frac{\prod_{k=1}^{M}(z - z_k)}{\prod_{k=1}^{N}(z - p_k)}$$

having complex conjugate poles (p_k) and zeros (z_k) :

A discrete-time system will be stable only if all of the poles of its transfer function $H(z)$ lie within the unit circle on the z-plane.

4 The Frequency Response of Discrete-Time Systems

Consider the response of the system $H(z)$ to an infinite complex exponential sequence

$$f_n = A\,\mathrm{e}^{\mathrm{j}\,\omega n} = A\cos(\omega n) + jA\sin(\omega n),$$

where ω is the normalized frequency (rad/sample). The response will be given by the convolution

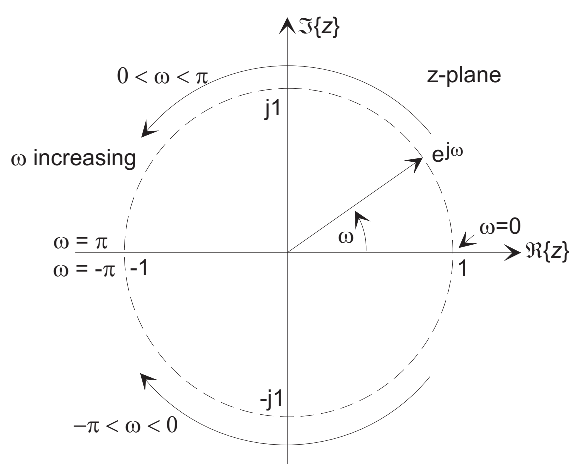
$$\begin{aligned} y_n &= \sum_{k=-\infty}^{\infty} h_k f_{n-k} = \sum_{k=-\infty}^{\infty} h_k \left(A\,\mathrm{e}^{\mathrm{j}\,\omega(n-k)}\right) \\ &= A\left(\sum_{k=-\infty}^{\infty} h_k\,\mathrm{e}^{-\mathrm{j}\,\omega k}\right)\mathrm{e}^{\mathrm{j}\,\omega n} \\ &= AH(e^{\mathrm{j}\,\omega})e^{\mathrm{j}\,\omega n} \end{aligned}$$

where the *frequency response function* $H(e^{\mathrm{j}\,\omega})$ is

$$\boxed{H(e^{\mathrm{j}\,\omega}) = H(z)|_{z=e^{\mathrm{j}\,\omega}}}$$

that is

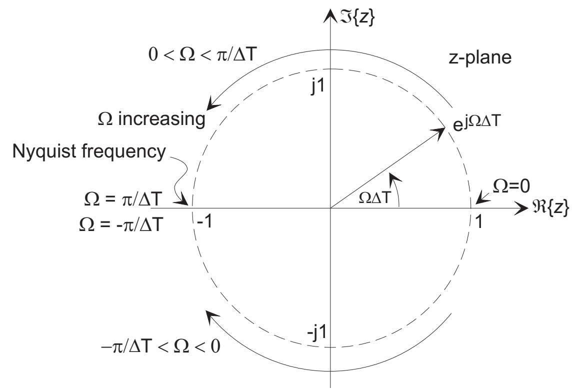
The frequency response function of a LTI discrete-time system is $H(z)$ evaluated on the unit circle - provided the ROC includes the unit circle. For a stable causal system this means there are no poles lying on the unit circle.



Alternatively, the frequency response may be based on a physical frequency Ω associated with an implied sampling interval ΔT, and

$$\boxed{H(e^{\mathrm{j}\,\Omega\Delta T}) = H(z)|_{z=e^{\mathrm{j}\,\Omega\Delta T}}}$$

which is again evaluated on the unit circle, but at angle $\Omega\Delta T$.

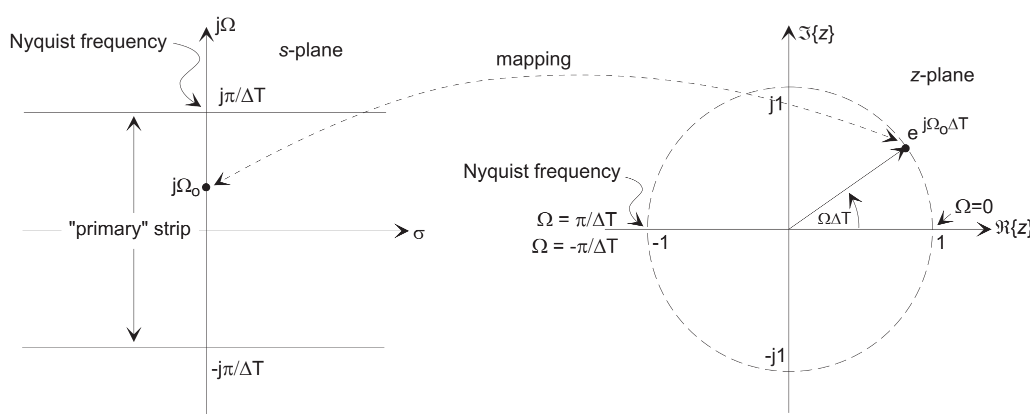


From the definition of the DTFT based on a sampling interval ΔT

$$H^*(j\Omega) = \sum_{n=0}^{\infty} h_n \,\mathrm{e}^{-mjn\Omega\Delta T} = H(z)|_{z=\,\mathrm{e}^{-mjn\Omega\Delta T}}$$

we can define the mapping between the imaginary axis in the s-plane and the unit-circle in the z-plane

$$s = \mathrm{j}\,\Omega_o \longleftrightarrow z = \mathrm{e}^{\mathrm{j}\,\Omega_o\Delta T}$$



The periodicity in $H(\,\mathrm{e}^{\mathrm{j}\,\Omega\Delta T})$ can be clearly seen, with the "primary" strip in the s-plane (defined by $-\pi/\Delta T < \Omega < \pi/\Delta T$) mapping to the complete unit-circle. Within the primary strip, the l.h. s-plane maps to the interior of the unit circle in the z-plane, while the r.h. s-plane maps to the exterior of the unit-circle.

Aside: We use the argument to differentiate between the various classes of transfer functions:

$H(s)$	$H(j\Omega)$	$H(z)$	$H(e^{j\omega})$
$\Updownarrow$	$\Updownarrow$	$\Updownarrow$	$\Updownarrow$
Continuous Transfer Function	Continuous Frequency Response	Discrete Transfer Function	Discrete Frequency Response

5 The Inverse z-Transform

The formal definition of the inverse z-transform is as a contour integral in the z-plane,

$$\frac{1}{2\pi j} \oint_{-\infty}^{\infty} F(z) z^{n-1} \, dz$$

where the path is a ccw contour enclosing all of the poles of $F(z)$.

Cauchy's residue theorem states

$$\frac{1}{2\pi j} \oint_{-\infty}^{\infty} F(z) \, dz = \sum_k \text{Res}\left[F(z), p_k\right]$$

where $F(z)$ has N distinct poles p_k, $k = 1, \ldots, N$ and ccw path lies in the ROC.
For a simple pole at $z = z_o$

$$\text{Res}\left[F(z), z_o\right] = \lim_{z \to z_o} (z - z_o) F(z),$$

and for a pole of multiplicity m at $z = z_o$

$$\text{Res}\left[F(z), z_o\right] = \lim_{z \to z_o} \frac{1}{(m-1)!} \frac{d^{m-1}}{dz^{m-1}} (z - z_o)^m F(z)$$

The inverse z-transform of $F(z)$ is therefore

$$\boxed{f_n = \mathcal{Z}^{-1}\left\{F(z)\right\} = \sum_k \text{Res}\left[F(z) z^{n-1}, p_k\right].}$$

■ Example 3

A first-order low-pass filter is implemented with the difference equation

$$y_n = 0.8 y_{n-1} + 0.2 f_n.$$

Find the response of this filter to the unit-step sequence $\{u_n\}$.

Solution: The filter has a transfer function

$$H(z) = \frac{Y(z)}{F(z)} = \frac{0.2}{1 - 0.8z^{-1}} = \frac{0.2z}{z - 0.8}$$

The input $\{f_n\} = \{u_n\}$ has a z-transform

$$F(z) = \frac{z}{z-1}$$

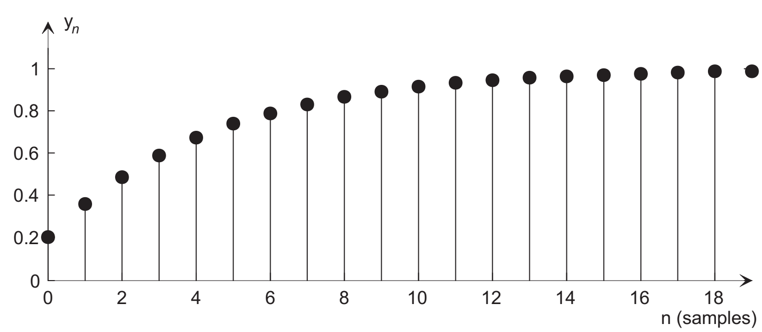
so that the z-transform of the output is

$$Y(z) = H(z)U(z) = \frac{0.2z^2}{(z-1)(z-0.8)}$$

and from the Cauchy residue theorem

$$\begin{aligned} y_n &= \mathrm{Res}\left[Y(z)z^{n-1}, 1\right] + \mathrm{Res}\left[Y(z)z^{n-1}, 0.8\right] \\ &= \lim_{z\to 1}(z-1)Y(z)z^{n-1} + \lim_{z\to 0.8}(z-0.8)Y(z)z^{n-1} \\ &= \lim_{z\to 1}\frac{0.2z^{n+1}}{z-0.8} + \lim_{z\to 0.8}\frac{0.2z^{n+1}}{z-1} \\ &= 1 - 0.8^{n+1} \end{aligned}$$

which is shown below



■ Example 4

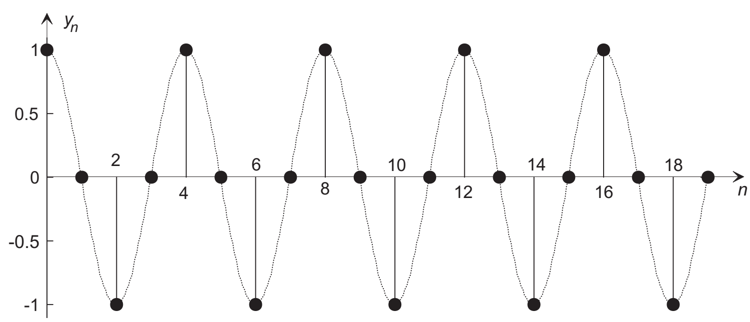
Find the impulse response of the system with transfer function

$$H(z) = \frac{1}{1+z^{-2}} = \frac{z^2}{z^2+1} = \frac{z^2}{(z+\mathrm{j}\,1)(z-\mathrm{j}\,1)}$$

Solution: The system has a pair of imaginary poles at $z = \pm \mathrm{j}\,1$. From the Cauchy residue theorem

$$\begin{aligned}
h_n &= \mathcal{Z}^{-1}\{H(z)\} = \mathrm{Res}\left[H(z)z^{n-1}, \mathrm{j}\,1\right] + \mathrm{Res}\left[H(z)z^{n-1}, -\mathrm{j}\,1\right] \\
&= \lim_{z \to j1} \frac{z^{n+1}}{z + \mathrm{j}\,1} + \lim_{z \to -\mathrm{j}\,1} \frac{z^{n+1}}{z - \mathrm{j}\,1} \\
&= \frac{1}{\mathrm{j}\,2}(\mathrm{j}\,1)^{n+1} - \frac{1}{\mathrm{j}\,2}(-\mathrm{j}\,1)^{n+1} \\
&= \frac{\mathrm{j}^{\,n}}{2}\left(1 + (-1)^{n+1}\right) \\
h_n &= \begin{cases} 0 & n \text{ odd} \\ (-1)^{n/2} & n \text{ even} \end{cases} \\
&= \cos(n\pi/2)
\end{aligned}$$

where we note that the system is a pure oscillator (poles on the unit circle) with a frequency of half the Nyquist frequency.



■ Example 5

Find the impulse response of the system with transfer function

$$H(z) = \frac{1}{1 + 2z + z^{-2}} = \frac{z^2}{z^2 + 2z + 1} = \frac{z^2}{(z+1)^2}$$

Solution: The system has a pair of coincident poles at $z = -1$. The residue at $z = -1$ must be computed using

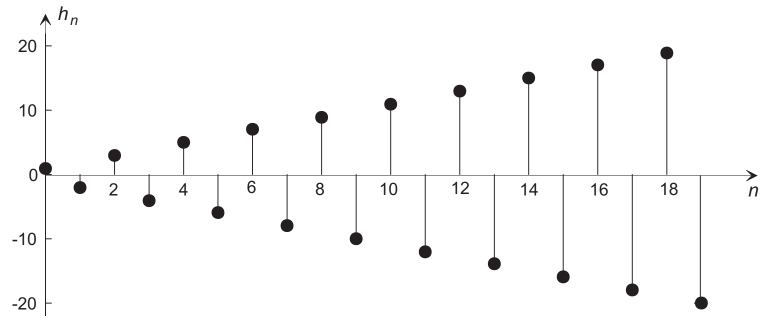
$$\mathrm{Res}\left[F(z), z_o\right] = \lim_{z \to z_o} \frac{1}{(m-1)!} \frac{\mathrm{d}^{m-1}}{\mathrm{d}z^{m-1}} (z - z_o)^m F(z).$$

With $m = 2$, at $z = -1$,

$$\begin{aligned}
\mathrm{Res}\left[H(z)z^{n-1}, -1\right] &= \lim_{z \to -1} \frac{1}{(1)!} \frac{\mathrm{d}}{\mathrm{d}z} (z-1)^2 H(z) z^{n-1} \\
&= \lim_{z \to -1} \frac{\mathrm{d}}{\mathrm{d}z} z^{n+1} \\
&= (n+1)(-1)^n
\end{aligned}$$

The impulse response is

$$h_n = \mathcal{Z}^{-1}\left\{H(z)\right\} = \text{Res}\left[H(z)z^{n-1}, -1\right] = (n+1)(-1)^n.$$



Other methods of determining the inverse z-transform include:

Partial Fraction Expansion: This is a table look-up method, similar to the method used for the inverse Laplace transform. Let $F(z)$ be written as a rational function of z^{-1}:

$$\begin{aligned} F(z) &= \frac{\sum_{k=0}^{M} b_i z^{-k}}{\sum_{k=0}^{N} a_i z^{-k}} \\ &= \frac{\prod_{k=1}^{M}(1 - c_i z^{-1})}{\prod_{k=1}^{N}(1 - d_i z^{-1})} \end{aligned}$$

If there are no repeated poles, $F(z)$ may be expressed as a set of partial fractions.

$$F(z) = \sum_{k=1}^{N} \frac{A_k}{1 - d_k z^{-1}}$$

where the A_k are given by the residues at the poles

$$A_k = \lim_{z \to d_k} (1 - d_k z^{-1})F(z).$$

Since

$$(d_k)^n\, u_n \quad \overset{Z}{\longleftrightarrow} \quad \frac{1}{1 - d_k z^{-1}}$$

$$\boxed{f_n = \left(\sum_{k=1}^{N} A_k\, (d_k)^n\right) u_n.}$$

■ Example 6

Find the response of the low-pass filter in Ex. 3 to an input

$$f_n = (-0.5)^n$$

Solution: From Ex. 3, and from the z-transform of $\{f_n\}$,

$$F(z) = \frac{1}{1-0.5z^{-1}}, \qquad H(z) = \frac{0.2}{1-0.8z^{-1}}$$

so that

$$\begin{aligned} Y(z) &= \frac{0.2}{(1+0.5z^{-1})(1-0.8z^{-1})} \\ &= \frac{A_1}{1+0.5z^{-1}} + \frac{A_2}{1-0.8z^{-1}} \end{aligned}$$

Using residues

$$\begin{aligned} A_1 &= \lim_{z\to-0.5} \frac{0.2}{1-0.8z^{-1}} = \frac{0.1}{1.3} \\ A_2 &= \lim_{z\to0.8} \frac{0.2}{1+0.5z^{-1}} = \frac{0.16}{1.3} \end{aligned}$$

and

$$\begin{aligned} y_n &= \frac{0.1}{1.3}\mathcal{Z}^{-1}\left\{\frac{1}{1+0.5z^{-1}}\right\} + \frac{0.16}{1.3}\mathcal{Z}^{-1}\left\{\frac{1}{1-0.8z^{-1}}\right\} \\ &= \frac{0.1}{1.3}(-0.5)^n + \frac{0.16}{1.3}(0.8)^n \end{aligned}$$

Note: (1) If $F(z)$ contains repeated poles, the partial fraction method must be extended as in the inverse Laplace transform.
(2) For complex conjugate poles – combine into second-order terms.

Power Series Expansion: Since

$$F(z) = \sum_{n=-\infty}^{\infty} f_n z^{-n}$$

if $F(z)$ can be expressed as a power series in z^{-1}, the coefficients must be f_n.

■ Example 7

Find $\mathcal{Z}^{-1}\{\log(1 + az^{-1})\}$.

Solution: $F(z)$ is recognized as having a power series expansion:

$$F(z) = \log(1 + az^{-1}) = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}a^n}{n} z^{-n} \qquad \text{for } |a| < |z|$$

Because the ROC defines a causal sequence, the samples f_n are

$$f_n = \begin{cases} 0 & \text{for } n \leq 0 \\ \dfrac{(-1)^{n+1}a^n}{n} & \text{for } n \geq 1. \end{cases}$$

Polynomial Long Division: For a causal system, with a transfer function written as a rational function, the first few terms in the sequence may sometimes be computed directly using polynomial division. If $F(z)$ is written as

$$F(z) = \frac{N(z^{-1})}{D(z^{-1})} = f_0 + f_1 z^{-1} + f_2 z^{-2} + f_2 z^{-2} + \cdots$$

the quotient is a power series in z^{-1} and the coefficients are the sample values.

■ Example 8

Determine the first few terms of f_n for

$$F(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}}$$

using polynomial long division.

Solution:

$$\begin{array}{r|l} & 1 + 4z^{-1} + 7z^{-2} + \cdots \\ \hline 1 - 2z^{-1} + z^{-2} & 1 + 2z^{-1} \\ & 1 - 2z^{-1} + z^{-2} \\ \hline & \quad 4z^{-1} - z^{-2} \\ & \quad 4z^{-1} - 8z^{-2} + 4z^{-3} \\ \hline & \qquad 7z^{-2} - 4z^{-3} \end{array}$$

so that

$$F(z) = \frac{1 + 2z^{-1}}{1 - 2z^{-1} + z^{-2}} = 1 + 4z^{-1} + 7z^{-2} + \cdots$$

and in this case the general term is

$$f_n = 3n + 1 \qquad \text{for } n \geq 0.$$

In general, the computation can become tedious, and it may be difficult to recognize the general term from the first few terms in the sequence.