2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.161 Signal Processing - Continuous and Discrete
Fall Term 2008

Problem Set 6 Solution

Assigned: October 23, 2008 **Due:** October 30, 2008

Problem 1:

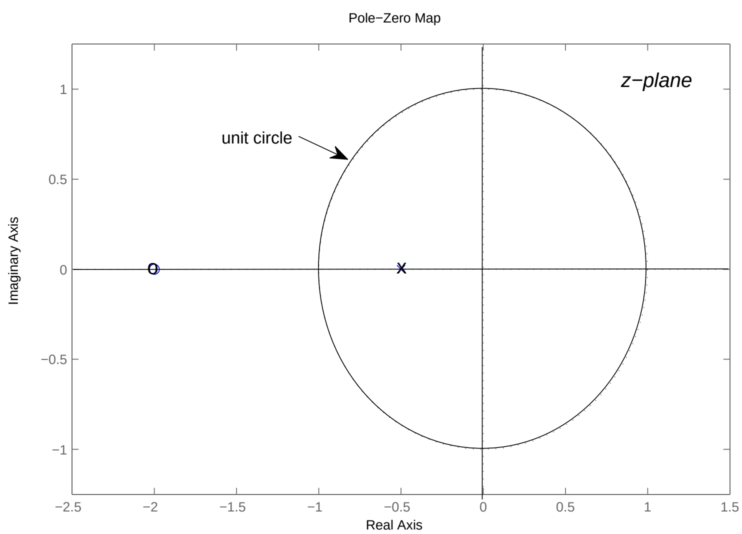
Given the difference equation,

$$y_n = -0.5y_{n-1} + 0.5u_n + u_{n-1}$$

(a) The transfer function is given by,

$$H(z) = \frac{Y(z)}{U(z)} = \frac{0.5 + 1z^{-1}}{1 + 0.5z^{-1}} = \frac{0.5z + 1}{z + 0.5}$$

(b) Pole zero map



(c) The system is causal, therefore the ROC includes all $\|z| > 0.5$ – which includes the unit circle. The system is therefore stable.

(d) Let $\Omega = \omega\Delta T$. The system frequency response magnitude is given by

$$\begin{aligned} \left|H(e^{j\Omega})\right| = \left|H(z)|_{z=e^{j\Omega}}\right| &= \frac{\left|0.5e^{j\Omega} + 1\right|}{|e^{j\Omega} + 0.5|} \\ &= \sqrt{\frac{1.25 + \cos(\Omega)}{1.25 + \cos(\Omega)}} \\ &= 1. \end{aligned}$$

and the system is an all-pass filter.
When $\omega = 0$ (or $\Omega = 0$), $\angle H(j\omega) = 0$.
When $\omega = \pi/T$ (or $\Omega = \pi$), $\angle H(j\omega) = -\pi$.

Problem 2:

For the following functions, we want a causal function, thus the ROC is $|z| > |\text{largest pole}|$. Since the poles are inside the unit circle, the functions are stable.

(a) Since $h_n = \mathcal{Z}^{-1}\{H_a(z)\}$, and

$$H_a(z) = \frac{1 - z^{-1}}{1 + 0.77z^{-1}} = \frac{1}{1 + 0.77z^{-1}} - \frac{z^{-1}}{1 + 0.77z^{-1}}$$

and from a table of z-transforms

$$h_n = (-0.77)^n u_n - (-0.77)^{n-1} u_{n-1}, \qquad n \geq 0$$

(b) Write the transfer function as

$$H_b(z) = \frac{z^2 + z}{z^2 + 0.9z + 0.81},$$

for $|z| > 0.9$. Then comparing with the given forms

$$\begin{aligned} \mathcal{Z}\{r^n \cos(an)\} &= \frac{z(z - r\cos(a))}{z^2 - 2r\cos(a)z + r^2} \\ \mathcal{Z}\{r^n \sin(an)\} &= \frac{r\sin(a)z}{z^2 - 2r\cos(a)z + r^2}, \end{aligned}$$

rewrite $H_b(z)$ as

$$H_b(z) = \frac{z^2 - r\cos(a)z}{z^2 - 2r\cos(a)z + r^2} + K\frac{r\sin(a)z}{z^2 - 2r\cos(a)z + r^2}.$$

where $-r\cos(a) + Kr\sin(a) = 1$, so that

$$h_n = (r^n \cos(an) + Kr^n \sin(an))\, u(n)$$

Comparing coefficients in the denominator

$$r = 0.9, \quad \cos(a) = -1/2, \quad \text{giving} \quad a = \frac{2}{3}\pi, \quad \sin(a) = \sqrt{3}/2, \quad \text{and} \quad K = \frac{1.1}{0.9\sqrt{3}}$$

or

$$h_n = \begin{cases} 0.9^n \left(\cos(2n\pi/3) + \frac{1.1}{0.9\sqrt{3}} \sin(2n\pi/3)\right) & n \geq 0 \\ 0 & n < 0 \end{cases}$$

Problem 3: Proakis and Manolakis: Problem 3.8 (p. 215)

(a)

$$y(n) = \sum_{k=-\infty}^{n} x(k) = \sum_{k=-\infty}^{\infty} x(k)u(n-k) = x(n) \otimes u(n)$$

$$Y(z) = X(z)U(z) = \frac{X(z)}{1-z^{-1}}$$

(b)

$$u(n) \otimes u(n) = \sum_{k=-\infty}^{\infty} u(k)u(n-k) = \sum_{k=-\infty}^{n} u(k) = (n+1)u(n)$$

$$X(z) = U(z)U(z) = \frac{1}{(1-z^{-1})^2}$$

Problem 4: Write

$$\begin{aligned} H(z) &= \frac{z^2}{z^2 - \frac{5}{6}z + \frac{1}{6}} \\ &= \frac{z^2}{(z-\frac{1}{2})(z-\frac{1}{3})} \\ &= \frac{3z}{z-\frac{1}{2}} - \frac{2z}{z-\frac{1}{3}} \\ &= \frac{3}{1-\frac{1}{2}z^{-1}} - \frac{2}{1-\frac{1}{3}z^{-1}} \end{aligned}$$

and

$$h_n = 3\left(\frac{1}{2}\right)^n u_n - 2\left(\frac{1}{3}\right)^n u_n$$

Alternatively, using MATLAB

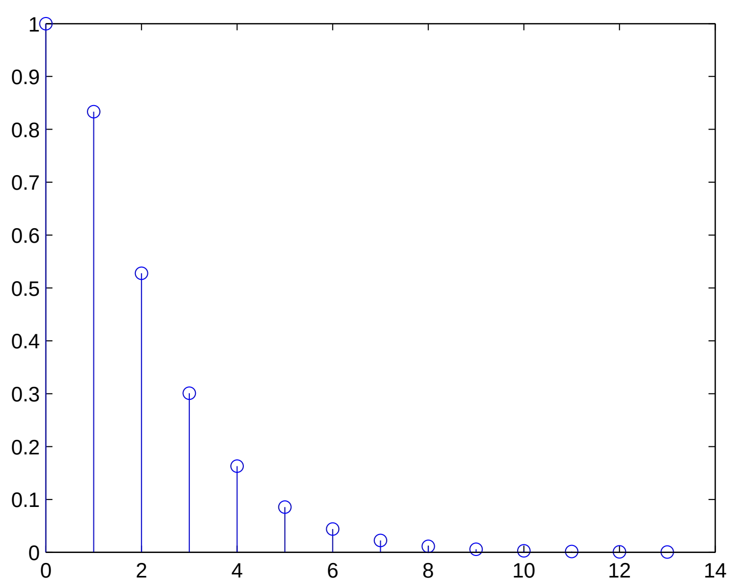
```
>> [r,p,k]=residuez([1 0 0],[1 -5/6 1/6])
r =
    2.99999999999999e+000
   -1.99999999999999e+000
p =
    500.000000000000e-003
    333.333333333333e-003
k =
    0.00000000000000e-003
>>
```

where p are the poles, and r are the residues at the poles. k contains the direct terms in a row vector (coefficients of $z^0, z^1, z^2, ...$ in the partial fraction expansion for the cases when numerator order is larger than denominator order).

The command

```
>> [h,t]=impz([1 0 0],[1 -5/6 1/6])
```

generates the following plot:



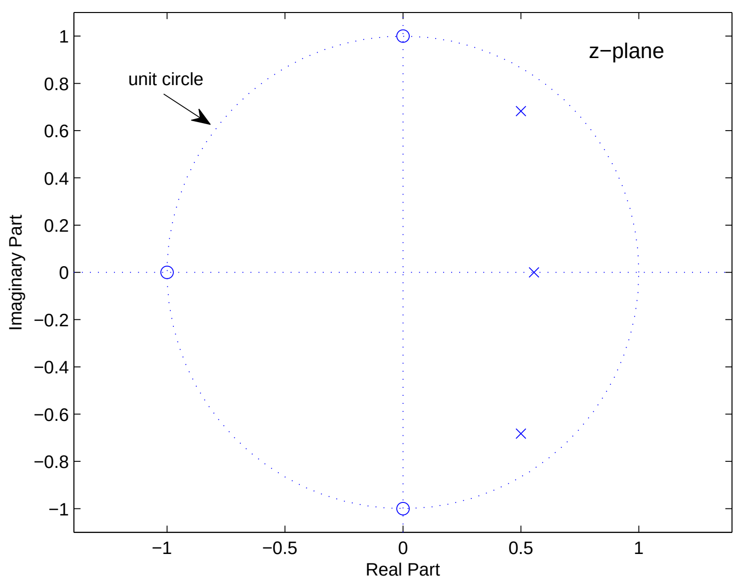
Problem 5:

(a) From $H(z)$

$$y_n = 1,556y_{n-1} - 1.272y_{n-2} + 0.398y_{n-3} + 0.0798(f_n + f_{n-1} + f_{n-2} + f_{n-3}).$$

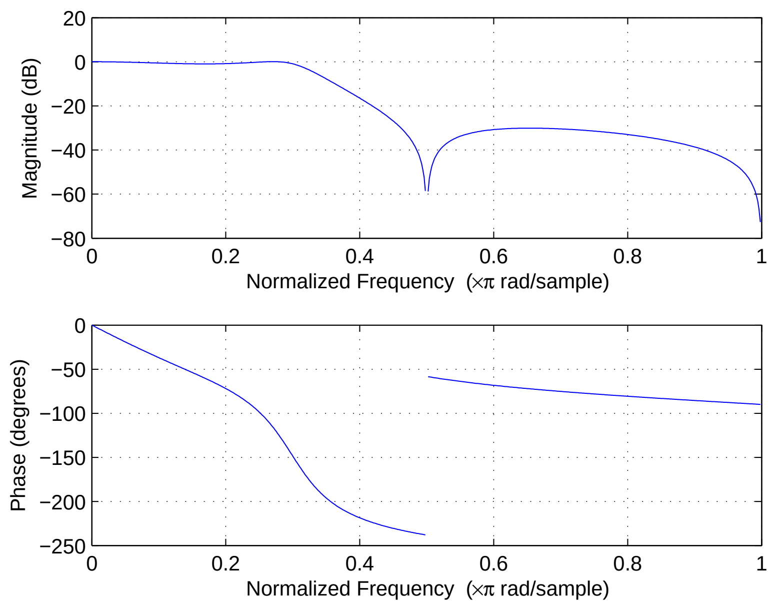
(b)
```
>> z=roots([1 1 1 1])
z =
   -1.00000000000000e+000
   -402.455846426619e-018 + 1.00000000000000e+000i
   -402.455846426619e-018 - 1.00000000000000e+000i
>> p=roots([1 -1.556 1.272 -0.398])
p =
    500.102320736184e-003 + 682.633555786812e-003i
    500.102320736184e-003 - 682.633555786812e-003i
    555.795358527632e-003
>> zplane(z,p)
>>
```

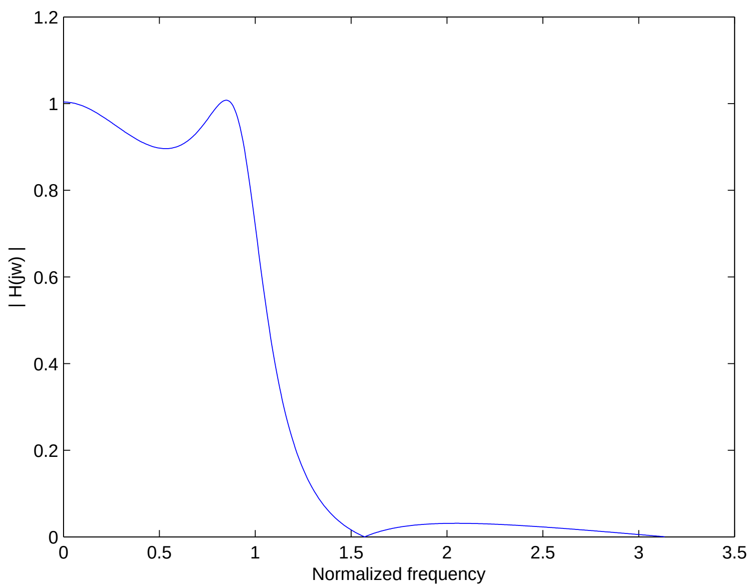
giving poles at 0.5558, $0.5001 \pm j0.6826$, and zeros at -1, $0 \pm j1.$, and the pole-zero plot:



(c)
```
>> a=[1 -1.556 1.272 -0.398];
>> b=0.0798*[1 1 1 1];
>> [H,w]=freqz(b,a);
>> figure
>> plot(w,abs(H));
>>
```

generates the following two plots (log-magnitude and linear-magnitude):





The pole-aero plot shows zeros on the unit-circle at angles $\Omega = \pi/2$ and π, indicating that the frequency response magnitude should dip to zero at these frequencies. This is seen on the frequency response plots. There are three poles, not on the unit-circle, but in the low-frequency region, indicating a low-pass action. Note the ripple in the pass-band and the stop-band - a characteristic of elliptic filters.

(d) The MATLAB function `[H,w]=freqz()` returns the frequency vector `w` normalized to the range $0 \leq \Omega \leq \pi$. The physical frequency ω is found from $\omega = \Omega/\Delta T$, where ΔT is the sampling interval. Experimentation with the data cursor on the linear magnitude plot finds that the -3 dB cut-off frequency is at $\Omega = 1$, giving the physical cut-off frequency $\omega = 1/10^{-4} = 10^4$ rad/s.