2.161 Signal Processing: Continuous and Discrete
Fall 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

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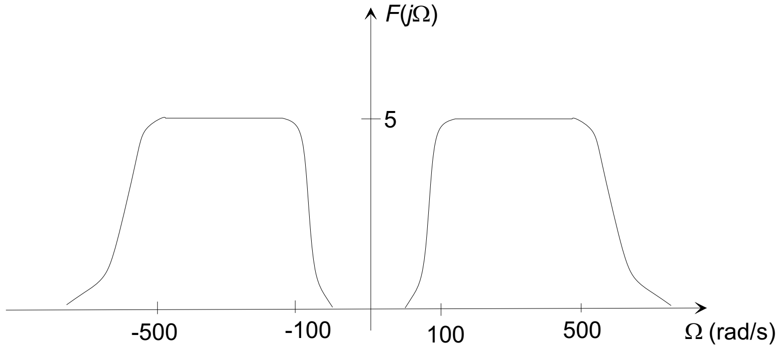
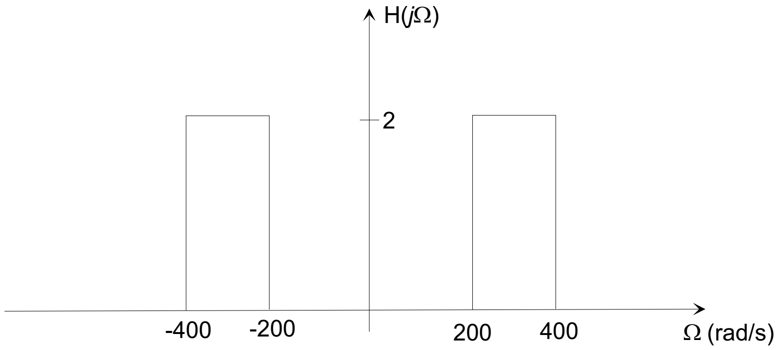
Problem Set 2

Assigned: Sept. 18, 2008 **Due:** Sept. 25, 2008

Problem 1: A waveform $f(t)$ with a real even spectrum $F(j\Omega)$

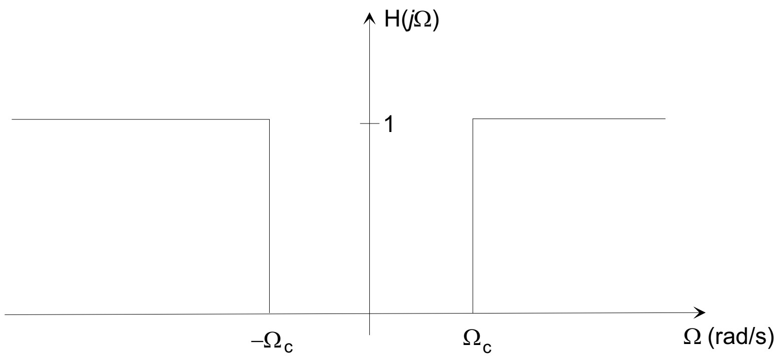


is filtered by an ideal band-pass filter with a purely real frequency response $H(j\Omega)$



Determine the filter response $y(t)$. Is this a causal filter?

Problem 2: Find the impulse response of an ideal high-pass filter, with cut-off frequency Ω_c:



Hints: Feel free to do this any way you wish, but you may find it useful to consider $H(j\Omega)$ as a superposition of functions with known FT relationships. In particular, the solution of Prob. 5 in PS 1 might be useful and/or the following bits and pieces from the Frequency Domain class handout: the Fourier transform of a unit step function (p. 32), the duality property, the time shift property, time reversal property, linearity property, all might help you.

Problem 3: Use the Fourier transform of a sinusoid (p. 34 of the class handout) to express the Fourier series representation of a periodic waveform

$$x(t) = \sum_{n=0}^{\infty} A_n \sin\left(n\Omega_0 t + \phi_n\right)$$

as a Fourier transform.

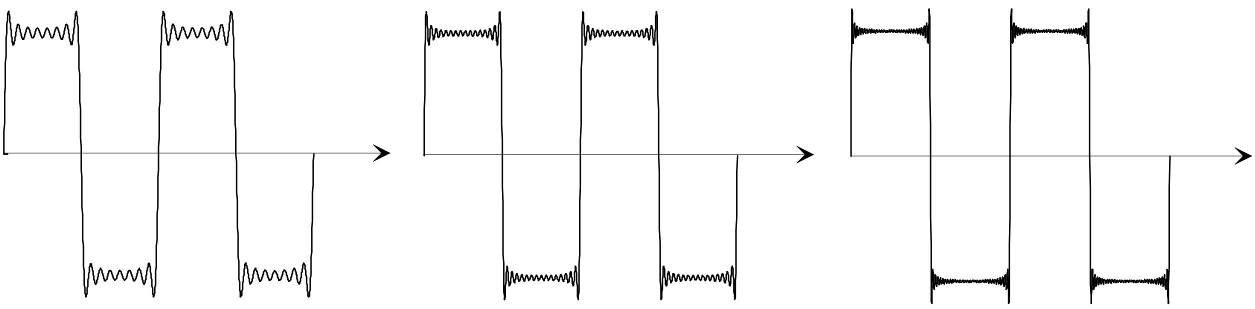
Problem 4: *Gibb's phenomenon* is associated with the synthesis of periodic waveforms with sharp (jump) discontinuities using a *truncated* Fourier series. Consider a periodic waveform $x_p(t)$ and its Fourier series

$$\begin{aligned} x_p(t) &= \sum_{n=-\infty}^{\infty} X_n e^{jn\Omega_0 t} \\ X_n &= \frac{1}{T}\int_{-T/2}^{T/2} x_p(t) e^{-jn\Omega_0 t} dt \end{aligned}$$

and let

$$\widehat{x}_{p,N}(t) = \sum_{n=-N}^{N} X_n e^{jn\Omega_0 t}$$

be an approximation generated by truncating the series. It was observed in the late 1800's that a finite series approximation created a *ripple* in the synthesized waveform in the region of a jump discontinuity in $x_p(t)$, and that although the width of the ripple decreased as more terms were included, the amplitude remained constant. The phenomenon was found to be present for any finite N. The figure below shows the synthesis of a square wave $\widehat{x}_{p,N}(t)$ with $N = 15, 31,$ and 63.



(a) N = 15　　(b) N = 31　　(c) N = 63

Gibb's phenomenon was the subject of much discussion in the mathematical literature around 1900, and in fact was cited as a reason why Fourier analysis/synthesis did not work! The great mathematicians of the time did not understand what was going on. You, however, do understand it and your task is to explain the phenomenon.

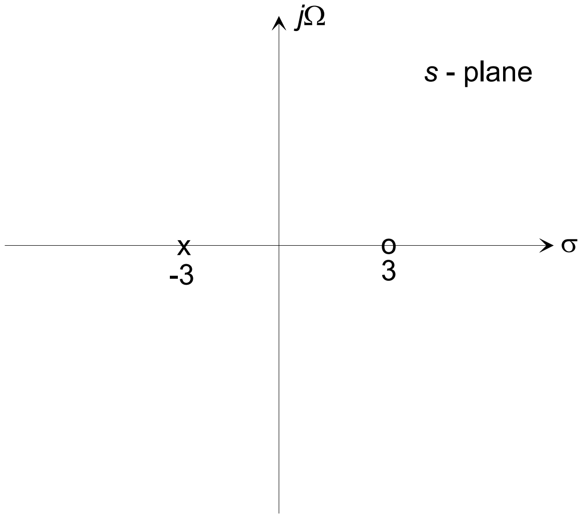
(a) Explain the truncation of the series in the frequency domain as an ideal multiplicative filtering operation, and define the filter pass-band.

(b) What is the equivalent time domain operation? OK - you win - it's convolution, but the question is what are the two functions being convolved? Write the appropriate convolution integral.

(c) Explain (i) why the height of the ripple on the square wave is constant for any finite series, and (ii) why the oscillatory frequency increases as N increases.

<u>**Problem 5:**</u> An "all-pass" filter has a frequency response magnitude $|H(j\Omega)|$ that is a constant (ie is independent of frequency Ω).

(a) Show that the first-order system with the pole-zero plot below is an all-pass filter.



(b) Show that any system with (i) the same number of poles and zeros, (ii) with all poles in the left-half s-plane, and (iii) the zeros mirror the poles about the imaginary axis, that is $z_i = -\overline{p}_i$ where $\overline{x}$ denotes the complex conjugate, is an all-pass filter.

(c) Use MATLAB to plot (i) the frequency response, (ii) the impulse response, and (iii) the step response of the filter with a pole and zero shown in (a) above. Choose an arbitrary gain constant.
(Some useful functions might be `zpk()`, `tf()`, `freqs()`, `bode()`, `impulse()`, `step()`.)

(d) What useful purpose might an all-pass filter serve?