

3 NONLINEAR COEFFICIENTS IN DETAIL

The method of hydrodynamic coefficients is a somewhat blind series expansion of the fluid force in an attempt to provide a framework on which to base experiments and calculations to evaluate these terms. The basic difficulty, i.e. the intractability of the governing equations of motion of viscous fluid prohibits, at least today and in the near future, a computation of these forces.

Still, we are not totally ignorant about these forces, since a number of symmetries and basic laws can be applied to reduce the number of unknown coefficients. This is the purpose of this section.

The basic assumption in using the method of hydrodynamic coefficients is that the forces have no memory effects, i.e. past motions have no impact on the fluid forces at the present moment. This is not correct when the flow separates, or when large vortices are shed, because then the vorticity in the fluid affects the fluid forces for a considerable time after they have been shed – i.e. until they move sufficiently far away from the body. We will show later some methods which allow us to incorporate the effect of shed vorticity, because under certain conditions such effects can not be ignored.

We employ the following basic facts and assumptions to derive the fluid forces acting on a ship, submarine or vehicle:

1. We retain only first order acceleration terms. Based on Newton's second law, we expect the inertia terms from the fluid to be linearly dependent on acceleration.
2. We do not include terms coupling velocities and accelerations. Again, based on Newton's second law, we expect inertia forces to depend on acceleration alone.
3. We consider port/starboard symmetry. Unless there is a reason not to, this is a useful property to us in order to eliminate a certain number of coefficients which are either zero or very small. In ships, a propeller introduces an asymmetry port/starboard since it rotates in a certain direction (unless it is a twin-screw ship with counter-rotating propellers), but in such cases we limit this asymmetry to only propeller-related terms.

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4. We retain up to third order terms. This is a practical consideration and has been found to, in general, serve well the purpose of deriving equations which are sufficiently accurate in a wide parametric range. It does not constitute an absolute rule, but most existing models employ this assumption

Finally, we find sometimes necessary to use coefficients such as $Y_{|v|v}$, providing a drag-related term, whose strict definition would be:

$$Y_{|v|v} = \frac{\partial^2 Y}{\partial v \partial |v|}(v = 0) \quad (36)$$

A Taylor series expansion would not include such terms, so their inclusion is motivated by physical arguments.

3.1 Helpful Facts

To exploit symmetries, we consider the following simple facts:

- A function $f(x)$ which is symmetric in x has zero odd derivatives with respect to x at $x = 0$. The proof is to consider the Taylor series expansion of $f(x)$ about $x = 0$:

$$f(x) = f(0) + \frac{df}{dx}(0)x + \frac{1}{2!} \frac{d^2 f}{dx^2}(0)x^2 + \frac{1}{3!} \frac{d^3 f}{dx^3}(0)x^3 + \dots \quad (37)$$

Symmetry gives that for all x :

$$f(x) = f(-x) \quad (38)$$

The only way that (38) is satisfied given the expression (37) is that all odd derivatives of $f(x)$ must be zero, i.e., for n odd:

$$\frac{d^n f}{dx^n}(0) = 0 \quad (39)$$

- A function $f(x)$ which is anti-symmetric in x has zero even derivatives with respect to x at $x = 0$. The proof is exactly analogous to the result above, using the anti-symmetry condition:

$$f(x) = -f(-x) \quad (40)$$

to find that, for n even:

$$\frac{d^n f}{dx^n}(0) = 0 \quad (41)$$

3.2 Nonlinear Equations in the Horizontal Plane

To demonstrate the methodology, we will derive the governing nonlinear equations of motion in the horizontal plane (surge, sway and yaw), employing the assumptions above. We will not include drag related terms of the form of equation 36 for the time being. We start by re-stating the inertia terms driven by external forces, which include the fluid forces:

$$\begin{aligned}
m(\dot{u} - vr - x_G r^2) &= X \\
m(\dot{v} + ur + x_G \dot{r}) &= Y \\
I_{zz} \dot{r} + mx_G(\dot{v} + ur) &= N
\end{aligned} \tag{42}$$

where we have assumed that $w = 0$, $p = 0$, $q = 0$, $y_G = 0$, $z_G = 0$.

3.2.1 Fluid Force X

By denoting the rudder angle as δ , we derive the following expression for the fluid force X , valid up to third order:

$$\begin{aligned}
X &= X_e + X_{\dot{u}} \dot{u} + X_u u + X_{uu} u^2 + X_{uuu} u^3 + X_{vv} v^2 + X_{rr} r^2 + X_{\delta\delta} \delta^2 + \\
&X_{rv} rv + X_{r\delta} r\delta + X_{v\delta} v\delta + X_{vvu} v^2 u + X_{rru} r^2 u + X_{\delta\delta u} \delta^2 u + \\
&X_{r\delta u} r\delta u + X_{rvu} rvu + X_{v\delta u} v\delta u + X_{rv\delta} rv\delta
\end{aligned} \tag{43}$$

We have used three basic properties, i.e., that the fluid force X , independent of the forward velocity, must be:

1. a symmetric function of v when $r = 0$ and $\delta = 0$;
2. a symmetric function of r when $v = 0$ and $\delta = 0$;
3. a symmetric function of δ when $r = 0$ and $v = 0$;

This is a result of port/starboard symmetry and is expressed as:

$$X(u, v, r = 0, \delta = 0) = X(u, -v, r = 0, \delta = 0) \tag{44}$$

$$X(u, v = 0, r, \delta = 0) = X(u, v = 0, -r, \delta = 0) \tag{45}$$

$$X(u, v = 0, r = 0, \delta) = X(u, v = 0, r = 0, -\delta) \tag{46}$$

These relations imply that all odd derivatives of X with respect to v at $v = 0$ are zero, when $r = 0$ and $\delta = 0$; and similarly for r and δ . For example:

$$\frac{\partial^3 X}{\partial v^3}(u, v = 0, r = 0, \delta = 0) = 0 \tag{47}$$

which implies that $X_{vvv} = 0$. Also, since relations such as (44) hold for all forward velocities u , it means that derivatives with respect to u of expression (44) are also true. For example, from (47) we derive:

$$\frac{\partial^4 X}{\partial v^3 \partial u}(u, v = 0, r = 0, \delta = 0) = 0 \quad (48)$$

or equivalently $X_{vvvu} = 0$.

In summary, the symmetries provide the following zero coefficients:

$$\begin{aligned} X_v &= 0; & X_{vvv} &= 0; & X_{vu} &= 0; & X_{vuu} &= 0; \\ X_r &= 0; & X_{rrr} &= 0; & X_{ru} &= 0; & X_{ruu} &= 0; \\ X_\delta &= 0; & X_{\delta\delta\delta} &= 0; & X_{\delta u} &= 0; & X_{\delta uu} &= 0 \end{aligned} \quad (49)$$

3.2.2 Fluid Force Y

In the case of the fluid force Y , the symmetry implies that this force must be an antisymmetric function of v when $r = 0$, $\delta = 0$; and likewise for r and δ , i.e.:

$$\begin{aligned} Y(u, v, r = 0, \delta = 0) &= -Y(u, -v, r = 0, \delta = 0) \\ Y(u, v = 0, r, \delta = 0) &= -Y(u, v = 0, -r, \delta = 0) \\ Y(u, v = 0, r = 0, \delta) &= -Y(u, v = 0, r = 0, -\delta) \end{aligned} \quad (50)$$

Hence, in analogy with the previous section, the even derivatives of Y with respect to v (and then r , and δ) must be zero, or:

$$\begin{aligned} Y_{vv} &= 0; & Y_{vvu} &= 0 \\ Y_{rr} &= 0; & Y_{rru} &= 0 \\ Y_{\delta\delta} &= 0; & Y_{\delta\delta u} &= 0 \end{aligned} \quad (51)$$

Finally, due to port/starboard symmetry, the force Y should not be affected by u when $v = 0$, $r = 0$, $\delta = 0$, except for propeller effects, which break the symmetry. For this reason, we will include terms such as Y_u to allow for the propeller asymmetry (twin propeller ships with counter-rotating propellers have zero such terms).

Finally, we derive the following expansion for Y :

$$\begin{aligned} Y &= Y_e + Y_u u + Y_{uu} u^2 + Y_{\dot{v}} \dot{v} + Y_{\dot{r}} \dot{r} + Y_v v + Y_r r + Y_\delta \delta + Y_{\delta u} \delta u + Y_{vu} v u + \\ &Y_{ru} r u + Y_{vuu} v u^2 + Y_{ruu} r u^2 + Y_{\delta uu} \delta u^2 + Y_{vvv} v^3 + Y_{rrr} r^3 + Y_{\delta\delta\delta} \delta^3 + \\ &Y_{rr\delta} r^2 \delta + Y_{rrv} r^2 v + Y_{vvr} v^2 r + Y_{vv\delta} v^2 \delta + Y_{vr\delta} v r \delta + Y_{\delta\delta r} \delta^2 r + Y_{\delta\delta v} \delta^2 v \end{aligned} \quad (52)$$

3.2.3 Fluid Moment N

The derivation for N follows the same exact steps as for the side force Y , i.e. the same symmetries apply. As a result, we first find that the following coefficients are zero:

$$\begin{aligned}
 N_{vv} &= 0; & N_{vvu} &= 0 \\
 N_{rr} &= 0; & N_{rru} &= 0 \\
 N_{\delta\delta} &= 0; & N_{\delta\delta u} &= 0
 \end{aligned} \tag{53}$$

Hence, we derive an analogous expansion for the moment:

$$\begin{aligned}
 N = & N_e + N_u u + N_{uu} u^2 + N_{\dot{\delta}} \dot{v} + N_{\dot{r}} \dot{r} + N_v v + N_r r + N_{\delta} \delta + N_{\delta u} \delta u + \\
 & N_{vu} v u + N_{ru} r u + N_{vuu} v u^2 + N_{ruu} r u^2 + N_{\delta uu} \delta u^2 + N_{vvv} v^3 + N_{rrr} r^3 + \\
 & N_{\delta\delta\delta} \delta^3 + N_{rr\delta} r^2 \delta + N_{rrv} r^2 v + N_{vvr} v^2 r + N_{vv\delta} v^2 \delta + N_{vr\delta} v r \delta + \\
 & N_{\delta\delta r} \delta^2 r + N_{\delta\delta v} \delta^2 v.
 \end{aligned} \tag{54}$$