

Generalized Energy Variables

Energetic interactions are mediated by the flow of power. Power flow through an interaction port may be expressed as the product of two real-valued variables, an effort and a flow, and all instantaneous interactions between systems or elements may be described in terms of these conjugate power variables.

However, to define the energy stored in a system (i.e. its instantaneous energetic state) it is necessary to define energy variables. Just as we may define two power variables, we may define two dual or conjugate energy variables, obtained by integrating the power variables with respect to time.

The first of these is *generalized momentum*¹, p .

$$p \triangleq \int_{t_0}^t e(t)dt + p(t_0) \quad (4.1)$$

It will be associated with *kinetic energy storage*. This relation may be differentiated.

$$dp = e dt \quad (4.2)$$

The conjugate variable is *generalized displacement*, q .

$$q \triangleq \int_{t_0}^t f(t)dt + q(t_0) \quad (4.3)$$

It will be associated with *potential energy storage*. This relation, too, may be differentiated.

$$dq = f dt \quad (4.4)$$

The following tables provide a partial list of energy variables and notation we will use for several energetic media.

¹ Sometimes known as *impulse*.

Table 4.1

Generalized Momenta and Notation

ENERGY MEDIUM	EFFORT	SYMBOL	MOMENTUM	SYMBOL
General		e		p
Mechanical translation	force	F	momentum or impulse	p
Fixed-axis mechanical rotation	torque or moment	τ (or μ)	angular momentum	η
Electrical	voltage or potential difference	e (or v)	flux linkage ²	λ
Magnetic	magnetomotive force	F	not defined	
Incompressible fluid flow	pressure difference	P	pressure momentum	Γ
Compressible fluid flow	enthalpy	h	not defined	
Thermal	temperature	θ (or T)	not defined	

Note that generalized momentum is not defined in some media. The reason is because there is no known kinetic energy storage phenomenon in those media. We will return to this point in a subsequently.

² Strictly speaking, this variable should be associated with displacement in the magnetic medium.

Table 4.2

Generalized Displacements and Notation

ENERGY MEDIUM	FLOW	SYMBOL	DISPLACEMENT	SYMBOL
General		f		q
Mechanical translation	speed or velocity	v	position or deflection	x
Fixed-axis mechanical rotation	angular speed or velocity	ω (or Ω)	angle	θ
Electrical	current	i	charge	q
Magnetic	flux rate	$\dot{\phi}$	flux	ϕ
Incompressible fluid flow	volumetric flow rate	Q	volume	V
Compressible fluid flow	mass flow rate	\dot{m}	mass	m
Thermal	entropy flow rate	\dot{s}	entropy	s,

Ideal Energy-Storage Elements

We are now in a position to define *ideal energy-storage elements*. (Ideal in the sense of not being contaminated by dissipation or any other non-storage phenomenon). The energy in a system may be determined from the power flux across its boundaries³.

$$E = \int_{t_0}^t P dt + E(t_0) \quad (4.5)$$

Using equations 4.2 and 4.4 and the definition of effort and flow, this may be rewritten in the following ways.

³ Once again we assume the convention that power is positive inwards.

$$E - E(t_0) = \int_{t_0}^t e f dt = \int_{t_0}^t e dq = \int_{t_0}^t f dp \quad (4.6)$$

Now, if we encounter a phenomenon characterized by a relation which permits the integral in either of the latter two forms to be evaluated so that it is not an explicit function of time, that phenomenon may be regarded as energy storage. As you might expect, there are two possibilities.

Generalized Capacitor

A *ideal generalized capacitor* is defined as any phenomenon characterized by an algebraic relation (possibly nonlinear) for which effort is an integrable (single-valued) function of displacement.

$$e = \Phi(q) \quad (4.7)$$

The algebraic function $\Phi(\cdot)$ is the constitutive equation for this element. Note that although we will use energy storage elements to describe dynamic behavior, this constitutive equation is a static or *memory-less* function. The constitutive equation permits us to evaluate the *generalized potential energy*, E_p

$$E_p \triangleq \int e dq = \int \Phi(q) dq = E_p(q) \quad (4.8)$$

For this element, potential energy is a function of displacement alone. It is a generalized potential energy storage element. The displacement, q , plays the same role as the specific entropy and specific volume do for a pure thermodynamic substance: it is sufficient to define the energy in the system. By convention we will define $E_p = 0$ at $q = 0$ as shown in figure 4.1.

It will turn out to be important to distinguish potential energy from a related quantity, (generalized) *potential co-energy*, E_p^* , which is a function of effort.

$$E_p^* \triangleq \int q de = E_p^*(e) \quad (4.9)$$

Energy and co-energy are related by a *Legendre transformation*:

$$E_p^*(e) = e q - E_p(q) \quad (4.10)$$

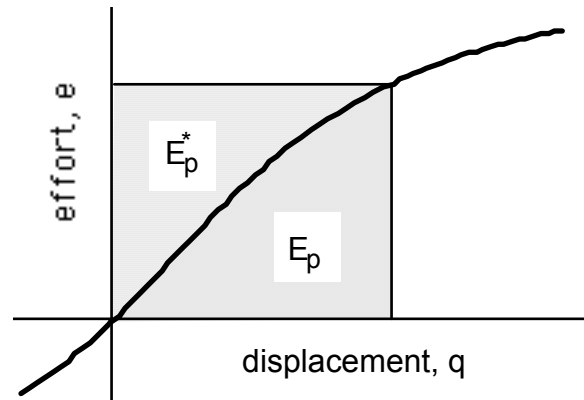


Figure 4.1: Sketch of a possible potential energy storage constitutive equation.

The relation between the two quantities is illustrated in figure 4.1. We will postpone further discussion of co-energy until later.

Taken together, the definitions of generalized displacement and the constitutive equation for a generalized capacitor specify a set of relations between flow, displacement and effort. These are represented by the symbol shown in figure 4.2. Note that as this is a passive element, power flow has been depicted as positive into it.



Figure 4.2: Bond graph symbol for an ideal capacitor.

An ideal capacitor may have a nonlinear constitutive relation. We may also define an *ideal linear* capacitor, one with a linear constitutive relation.

$$e = q/C \tag{4.11}$$

The parameter C is termed the capacitance of this ideal linear element. The potential energy stored in an ideal linear capacitor is a quadratic function of displacement.

$$E_p = q^2/2C \tag{4.12}$$

Aside: The reason for writing equation 4.11 with the proportionality constant, C , dividing the argument is largely historical. The generalized capacitor is based on an electrical capacitor, usually described by a linear relation between charge, q , (displacement) and voltage, e , (effort).

$$q = C e \tag{4.13}$$

However, in the nonlinear case it may not always be possible to express charge as a function of voltage; the fundamental definition is the one which permits the energy integral to be evaluated: voltage as a function of charge as in equation 4.7. To be consistent the fundamental definition yet acknowledge historical precedent, the parameter, C , has been written as in equation 4.11.

Another example of an ideal linear capacitor is the common model of a mechanical spring. If its deflections are small and occur at modest rates of change, it may be well described by Hooke's law:

$$F = k \Delta x \quad (4.14)$$

where F is force, k is stiffness and Δx is deflection. Equating deflection, Δx , with displacement, q , and force, F , with effort, e , this model provides a mechanical example of an ideal linear potential energy storage element with capacitance $1/k$. A bond graph symbol with the parameter included is shown in figure 4.3.

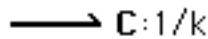


Figure 4.3: Bond graph symbol for an ideal linear potential energy storage element with capacitance $1/k$.

For large length changes, the force-deflection relation for typical mechanical spring departs from linear and the device provides a mechanical example of an ideal capacitor. In either case, the important point is that the stored potential energy is a function only of the displacement, q .

Generalized Inertia

In an exactly dual manner, an ideal *generalized inertia* is defined as any phenomenon characterized by an algebraic relation (possibly nonlinear) for which flow is an integrable (single-valued) function of momentum.

$$f = \Psi(p) \quad (4.15)$$

The algebraic function $\Psi(\cdot)$ is the constitutive equation for this element. Again, it is a static or memory-less function. This constitutive equation permits us to evaluate the *generalized kinetic energy*, E_k .

$$E_k \triangleq \int f dp = \int \Psi(p) dp = E_k(p) \quad (4.16)$$

For this element the potential energy is a function of the momentum alone. It is a generalized kinetic energy storage element. By convention we will define $E_k = 0$ at $p = 0$ as shown in figure 4.5.

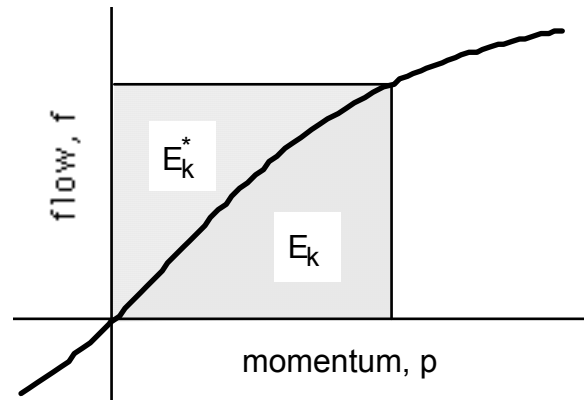


Figure 4.4: Sketch of a possible kinetic energy storage constitutive equation.

It will turn out to be important to distinguish kinetic energy from a related quantity, (generalized) *kinetic co-energy*, E_k^* , which is a function of flow.

$$E_k^* \triangleq \int p \, df = E_k^*(f) \quad (4.17)$$

Once again, energy and co-energy are related by a Legendre transformation:

$$E_k^*(f) = f p - E_k(p) \quad (4.18)$$

The relation between the two quantities is illustrated in figure 4.4.

Taken together, the definitions of generalized momentum and the constitutive equation for a generalized inertia specify a set of relations between effort, momentum and flow. These may be represented by the symbol shown in figure 4.5. Again, as this is a passive element, power flow has been depicted as positive into it.



Figure 4.5: Bond graph symbol for an ideal inertia.

An ideal inertia may have a nonlinear constitutive relation. We may also define an *ideal linear* inertia, one with a linear constitutive relation.

$$f = p/I \quad (4.19)$$

The parameter I is termed the inertance of this ideal linear element. The kinetic energy stored in an ideal linear inertia is a quadratic function of momentum.

$$E_k = p^2/2I \quad (4.20)$$

A common mechanical example of an ideal linear kinetic energy storage element is a body in motion. If the deflections of the body are small enough that it may be regarded as rigid, it may be characterized by a linear relation between velocity and momentum.

$$v = p/m \tag{4.21}$$

where v is velocity (flow), p is momentum and m is mass (inertance). A bond graph symbol with the parameter included is shown in figure 4.6.



Figure 4.6: Bond graph symbol for an ideal linear kinetic energy storage element with inertance m .

Once again, the reason for writing equation 4.19 with the proportionality constant, I , dividing the argument is to be consistent with historical precedent. Equation 4.21 is frequently written with momentum as function of velocity as follows.

$$p = m v \tag{4.22}$$

However, in a general, nonlinear case it may not be possible to express generalized momentum as a function of effort and the fundamental definition is the one which permits the energy integral to be evaluated: flow as a function of momentum as in equation 4.15.

Another practice sustained by historical precedent is to equate kinetic energy with the integral of equation 4.22 with respect to velocity. This is regrettable as this quantity is properly termed kinetic co-energy.

$$E_k^* = m v^2/2 \tag{4.23}$$

At this point we do no more than note the point of confusion and proceed. The importance of the distinction between energy and co-energy will be discussed in depth later.

Modulated Energy Storage is Prohibited

Previously we encountered the use of modulated power sources to describe how a control system might influence the energy supplied to or removed from a system. When we consider energy-storage elements, an important restriction must be emphasized: *modulation of energy storage elements is prohibited.*

The reason for this restriction is that a modulated energy-storage element would mean that the total energy in a system would be a function of the modulating input or set of inputs. Consequently, the total energy in the system would not be equal to the net power flow in across the system boundaries.. The system equations would not be guaranteed to satisfy the first law of thermodynamics; but that was to be the objective of our energy-based formalism.

The basic idea behind lumped-parameter modelling is to identify different phenomena with different elements. To be consistent with the energy-based formalism, the function of adding or removing energy from a system must be represented only by elements defined for that purpose: power sources and dissipators. The point to remember is: power sources and dissipators may be modulated; energy-storage elements may not.