2.092/2.093 — Finite Element Analysis of Solids & Fluids I Fall '09

Lecture 23 - Solution of $\boldsymbol{K\phi} = \lambda \boldsymbol{M\phi}$

Prof. K. J. Bathe MIT OpenCourseWare

Reading assignment: Chapters 10, 11

We have the solutions $0 < \underbrace{\lambda_1}_{\boldsymbol{\phi}_1} \leq \underbrace{\lambda_2}_{\boldsymbol{\phi}_2} \leq \ldots \leq \underbrace{\lambda_n}_{\boldsymbol{\phi}_n}$. Recall that:

$$\boldsymbol{K\phi}_i = \lambda_i \boldsymbol{M\phi}_i \tag{1}$$

In summary, a necessary and sufficient condition for $\boldsymbol{\phi}_i$ is that Eq. (1) is satisfied. The orthogonality conditions are **not** sufficient, unless $q = n$. In other words, vectors exist which are $\boldsymbol{K}$- and $\boldsymbol{M}$-orthogonal, but are not eigenvectors of the problem.

$$\boldsymbol{\Phi} = \begin{bmatrix} \boldsymbol{\phi}_1 & \ldots & \boldsymbol{\phi}_n \end{bmatrix} \tag{2}$$

$$\boldsymbol{\Phi^T M \Phi} = \boldsymbol{I} \quad ; \quad \boldsymbol{\Phi^T K \Phi} = \boldsymbol{\Lambda} = \begin{bmatrix} \lambda_1 & & \text{zeros} \\ & \ddots & \\ \text{zeros} & & \lambda_n \end{bmatrix} \tag{3}$$

Assume we have an $n \times q$ matrix $\boldsymbol{P}$ which gives us

$$\boldsymbol{P^T M P} = \underset{q \times q}{\boldsymbol{I}} \quad ; \quad \boldsymbol{P^T K P} = \underset{q \times q}{\boldsymbol{A}} \rightarrow \text{ diagonal matrix}$$

Is a_{ii} necessarily equal to λ_i?

$$\begin{bmatrix} a_{11} & & \text{zeros} \\ & a_{22} & \\ \text{zeros} & & \ddots \end{bmatrix}$$

If $q = n$, then $\boldsymbol{A} = \boldsymbol{\Lambda}$, $\boldsymbol{P} = \boldsymbol{\Phi}$ with some need for rearranging. If $q < n$, then $\boldsymbol{P}$ *may* contain eigenvectors (but not necessarily), and $\boldsymbol{A}$ *may* contain eigenvalues.

Rayleigh-Ritz Method

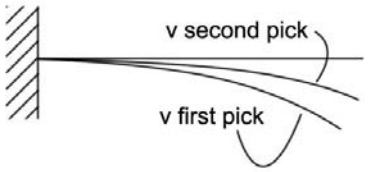
This method is used to calculate approximate eigenvalues and eigenvectors.

$$\rho(\boldsymbol{v}) = \frac{\boldsymbol{v}^T \boldsymbol{K v}}{\boldsymbol{v}^T \boldsymbol{M v}}$$

$$\lambda_1 \leq \rho(\boldsymbol{v}) \leq \lambda_n$$

λ_1 is the lowest eigenvalue, and λ_n is the highest eigenvalue of the system. λ_1 is related to the least strain energy that can be stored with $\boldsymbol{v}^T \boldsymbol{M v} = 1$:

$$\boldsymbol{\phi}_1^T \boldsymbol{K \phi}_1 = \lambda_1 \quad (\text{if } \boldsymbol{\phi}_1^T \boldsymbol{M \phi}_1 = 1)$$



Note that twice the strain energy is obtained when the system is subjected to $\boldsymbol{\phi}_1$. If the second pick for $\boldsymbol{v}$ gives a smaller value of $\rho(\boldsymbol{v})$, then the second pick is a better approximation to $\boldsymbol{\phi}_1$.

Assume $\overline{\boldsymbol{\phi}} = \sum_{i=1}^{q} \boldsymbol{\psi}_i x_i$, and the Ritz vectors $\boldsymbol{\psi}_i$ are linearly independent. Also, $\boldsymbol{\Psi} = [\,\boldsymbol{\psi}_1 \ldots \boldsymbol{\psi}_q\,]$. The x_i will be selected to minimize $\rho(\overline{\boldsymbol{\phi}})$. Hence, calculate $\frac{\partial}{\partial x_i}\rho(\overline{\boldsymbol{\phi}}) = 0$. (See Chapter 10.) The result is

$$\tilde{\boldsymbol{K}}\boldsymbol{x} = \rho \tilde{\boldsymbol{M}}\boldsymbol{x} \tag{4}$$

$$\tilde{\boldsymbol{K}} = \boldsymbol{\Psi}^{\boldsymbol{T}}\boldsymbol{K}\boldsymbol{\Psi} \quad ; \quad \tilde{\boldsymbol{M}} = \boldsymbol{\Psi}^{\boldsymbol{T}}\boldsymbol{M}\boldsymbol{\Psi} \tag{5}$$

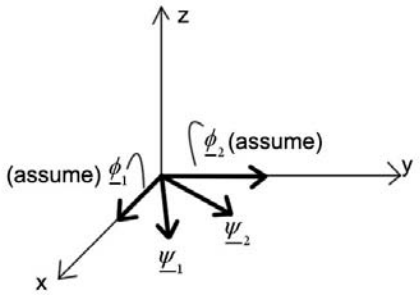
We solve Eq. (4) to obtain $\rho_1, \rho_2, \ldots, \rho_q$ and $\boldsymbol{x}_1, \boldsymbol{x}_2, \ldots, \boldsymbol{x}_q$. Then our approximation to $\lambda_1, \ldots, \lambda_q$ is given by $\rho_1, \ldots, \rho_q$.

$$\lambda_1 \le \rho_1 \quad ; \quad \lambda_2 \le \rho_2 \quad ; \quad \lambda_q \le \rho_q$$

$$\overline{\boldsymbol{\phi}}_1 \approx \boldsymbol{\phi}_1 \quad ; \quad \overline{\boldsymbol{\phi}}_2 \approx \boldsymbol{\phi}_2 \quad ; \quad \text{etc.}$$

where $\underset{n\times q}{[\,\overline{\boldsymbol{\phi}}_1 \ldots \overline{\boldsymbol{\phi}}_q\,]} = \underset{n\times q}{\boldsymbol{\Psi}}\,\underset{q\times q}{[\,\boldsymbol{x}_1 \ldots \boldsymbol{x}_q\,]}$.

If the q Ritz vectors span the subspace given by $\boldsymbol{\phi}_1, \ldots, \boldsymbol{\phi}_q$, then we obtain $(\lambda_1 \ldots \lambda_q)$ and $(\boldsymbol{\phi}_1 \ldots \boldsymbol{\phi}_q)$. Pictorially, an example:



If $\boldsymbol{\psi}_1$ and $\boldsymbol{\psi}_2$ are in the x-y plane, then by the Rayleigh-Ritz analysis we get $\boldsymbol{\phi}_1$, $\boldsymbol{\phi}_2$. Major shortcoming: in general, we do not know the accuracy of $(\rho_i, \overline{\boldsymbol{\phi}}_i)$.

The Subspace Iteration Method

Pick $\underset{n\times q}{\boldsymbol{X}_1}$, then calculate for $k = 1, 2, 3, \ldots$

$$\boldsymbol{K}\overline{\boldsymbol{X}}_{k+1} = \boldsymbol{M}\boldsymbol{X}_k \tag{a}$$

This is inverse iteration with q vectors. Now perform the Rayleigh-Ritz solution:

$$\boldsymbol{K}_{k+1} = \overline{\boldsymbol{X}}_{k+1}^T \boldsymbol{K}\overline{\boldsymbol{X}}_{k+1} \quad ; \quad \boldsymbol{M}_{k+1} = \overline{\boldsymbol{X}}_{k+1}^T \boldsymbol{M}\overline{\boldsymbol{X}}_{k+1} \tag{b}$$

$$\boldsymbol{K}_{k+1}\boldsymbol{Q}_{k+1} = \boldsymbol{M}_{k+1}\boldsymbol{Q}_{k+1}\boldsymbol{\Lambda}_{k+1} \tag{c}$$

$\boldsymbol{K}_{k+1}$, $\boldsymbol{M}_{k+1}$, and $\boldsymbol{Q}_{k+1}$ have dimensions $q \times q$. Recall that we have $\boldsymbol{K\Phi} = \boldsymbol{M\Phi\Lambda}$ from Eq. (1). We then have

$$\boldsymbol{Q}_{k+1}^T \boldsymbol{K}_{k+1}\boldsymbol{Q}_{k+1} = \boldsymbol{\Lambda}_{k+1} \quad ; \quad \boldsymbol{Q}_{k+1}^T \boldsymbol{M}_{k+1}\boldsymbol{Q}_{k+1} = \boldsymbol{I} \tag{d}$$

Finally,

$$\boldsymbol{X}_{k+1} = \overline{\boldsymbol{X}}_{k+1}\boldsymbol{Q}_{k+1} \tag{e}$$

Equations (b), (c), and (e) correspond to the use of the Rayleigh-Ritz method.

Then, provided the vectors in $\boldsymbol{X}_1$ are not $\boldsymbol{M}$-orthogonal to the eigenvectors we seek, we have (with "good" ordering) that

$$\boldsymbol{\Lambda}_{k+1} \to \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_q \end{bmatrix}$$

$$\boldsymbol{X}_{k+1} \to \begin{bmatrix} \boldsymbol{\phi}_1 & \dots & \boldsymbol{\phi}_q \end{bmatrix}$$

In practice, we use q vectors to calculate the p lowest eigenvalues, with (say) $q = 2p$. In fact, the convergence rate of the vectors is given by $\frac{\lambda_i}{\lambda_{q+1}}$.

If $p = 2$ and we have a multiplicity of 5 (or higher), $q = 2p$ corresponds to not enough vectors. Ideally, we want λ_{q+1} to be significantly larger than λ_p, so that $\frac{\lambda_i}{\lambda_{q+1}}$ is much less than 1 for $i = 1, \dots, p$. The "quite conservative" way is to use

$$q = \max(2p, p + 8)$$

The textbook gives $q = \min(2p, p+8)$, which can also be used (apply the Sturm sequence check, see textbook); it will use less storage, but will generally need more iterations. For modern computers (specifically with parallel processing), the above formula for q is frequently more effective.

Notice that $\boldsymbol{X}_{k+1}^T \boldsymbol{M} \boldsymbol{X}_{k+1} = \boldsymbol{I}$ because from (e),

$$\boldsymbol{Q}_{k+1}^T \underbrace{\overline{\boldsymbol{X}}_{k+1}^T \boldsymbol{M} \overline{\boldsymbol{X}}_{k+1}}_{\boldsymbol{M}_{k+1}} \boldsymbol{Q}_{k+1} = \boldsymbol{I}$$

2.092 / 2.093 Finite Element Analysis of Solids and Fluids I
Fall 2009