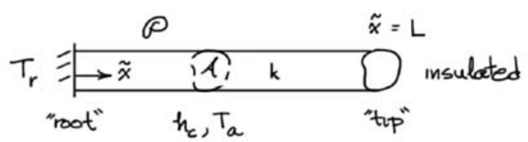
the Finite Element Method

a small introduction

a "fin" problem

ODE BVP

dimensional form



$$\frac{h_c(\mathcal{A}/\mathcal{P})}{k} \ll 1$$

$$(\Rightarrow T = T(x))$$

$$\begin{cases} -k\mathcal{A}\dfrac{d^2T}{d\tilde{x}^2} + h_c\mathcal{P}(T - T_a) = 0 & 0 < x < L \\ T(\tilde{x} = 0) = T_r \,, \quad \dfrac{dT}{dx}(\tilde{x} = L) = 0 & \end{cases}$$

non-dimensional form

Let

$$x = \tilde{x}/L \,, \quad \theta = \frac{T - T_a}{T_L - T_a} \,, \quad \mu = \frac{h_c \mathcal{P} L^2}{k\mathcal{A}} \;;$$

then

$$\begin{cases} -\dfrac{d^2\theta}{dx^2} + \mu\theta = 0 & 0 < x < 1 \\ \theta(x = 0) = 1 \,, \quad \dfrac{d\theta}{dx}(x = 1) = 0 & . \end{cases}$$

a convenient transformation

Let

$$\theta(x) = 1 + u(x) ;$$

then

$$\left\{ \begin{array}{l} -\dfrac{d^2u}{dx^2} + \mu u = \overbrace{-\mu}^{f} \qquad 0 < x < 1 \\[2ex] \underbrace{u(x=0) = 0}_{\text{bc1}} , \quad \underbrace{\dfrac{du}{dx}(x=1) = 0}_{\text{bc2}} \quad . \end{array} \right. \Bigg| \text{ TO SOLVE}$$

EXACT solution:

$$u(x) = \cosh(\sqrt{\mu}\,(1-x))/\cosh\sqrt{\mu} \; - 1 ;$$

$$\theta(x) = \cosh(\sqrt{\mu}\,(1-x))/\cosh\sqrt{\mu} \; .$$

a Minimization Statement

a "space" of admissible functions : X

We consider functions $w(x)$, $0 \le x \le 1$, such that:

$$\left. \begin{array}{l} \text{(i) } w(0) = 0 ; \text{ and} \\[1ex] \text{(ii) } w(x) \text{ is suitably smooth,} \\ \qquad \displaystyle\int_0^1 \left(\frac{dw}{dx}\right)^2 dx \text{ is } \underline{\text{finite}} . \end{array} \right\} \equiv X$$

a functional : or "energy"

Define, for any w (in X),

$$J(w) = \frac{1}{2} \int_0^1 \left(\frac{dw}{dx}\right)^2 + \mu w^2 - 2w\overset{-\mu}{f}\, dx ;$$

note

w (input) is a <u>function</u>

while

$J(w)$ (output) is a <u>real number</u>.

the minimization principle:

In equations:

$$\boxed{J(\underset{\text{solution of our ODE BVP}}{u}) < J(w) \quad \text{for any } w \text{ (in } X) \neq u .}$$

In words:

the solution of our ODE BVP, u,

is the minimizer of J over all functions in X.

proof: 1/6

Let

$$\underset{\text{any candidate in } X}{w} = \underset{\text{solution of our ODE BVP}}{u} + \underset{\text{a perturbation (not necessarily small)}}{v} .$$

Note since w is in X,

$$w(0) = 0 ;$$

but also

$$u(0) = 0 \text{ from boundary condition bc1 ;}$$

hence

$$\boxed{v(0) = 0 .} \quad \bigstar \quad v \text{ is in } X$$

2/6

Then,

$$J(u+v) = \frac{1}{2}\int_0^1 \frac{d}{dx}(u+v)\frac{d}{dx}(u+v) + \mu(u+v)(u+v) - 2(u+v)f\,dx$$

$$= \frac{1}{2}\int_0^1 \left(\frac{du}{dx}\right)^2 + \mu u^2 - 2uf\,dx \qquad J(u)$$

$$+ \int_0^1 \frac{dv}{dx}\frac{du}{dx} + \mu v u - 2vf\,dx$$

$$+ \frac{1}{2}\int_0^1 \left(\frac{dv}{dx}\right)^2 dx .$$

3/6

But, from integration by parts,

$$\int_0^1 \underset{\text{up}}{\frac{dv}{dx}}\,\underset{\text{down}}{\frac{du}{dx}}\,dx = v\frac{du}{dx}\bigg|_0^1 - \int_0^1 v\frac{d^2u}{dx^2}\,dx ;$$

furthermore,

$$v\frac{du}{dx}\bigg|_0^1 = \underset{\text{0 from bc2}}{v(1)\frac{du}{dx}(1)} - \underset{\text{0 from } \bigstar}{v(0)\frac{du}{dx}(0)}$$

$$= 0 \quad \text{for any } v \text{ in } X$$

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Hence,

$$\underline{\int_0^1 \frac{dv}{dx}\frac{du}{dx}} + \mu v u - v f \, dx$$

$$= \underline{\int_0^1 v\left(-\frac{d^2u}{dx^2}\right)} + \mu v u - v f \, dx$$

$$= \int_0^1 v \underbrace{\left\{ -\frac{d^2u}{dx^2} + \mu u - f \right\}}_{0 \text{ since } u \text{ is the solution to our ODE BVP}} dx$$

$= 0$ for any v in X.

5/6

Thus,

$$J(w) = J(u+v)$$

$$= J(u) + \frac{1}{2}\int_0^1 \left(\frac{dv}{dx}\right)^2 dx \quad \text{for any } v \text{ in } X.$$

($\Rightarrow$ any w in X)

But,

$$\int_0^1 \left(\frac{dv}{dx}\right)^2 dx > 0 \quad \text{unless } v(x) = \text{Const},$$

and since $v(0) = 0$ from ★ we conclude

$$\int_0^1 \left(\frac{dv}{dx}\right)^2 dx > 0 \quad \text{unless } v(x) = 0 .$$

6/6

In conclusion, for any $v(x)$ any $w = u + v$ in X

$$J(w) = J(u+v)$$

$$> J(u) \quad \text{unless } v(x) = 0 \iff w = u$$

$$\Downarrow$$

$$J(w) > J(u) \quad \text{for any } w \text{ in } X,\ w \neq u.$$

$$\Downarrow$$

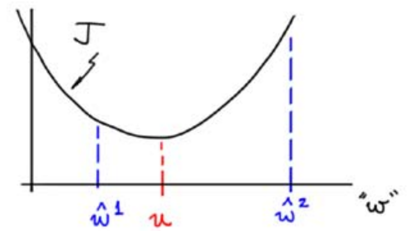
$$\boxed{u \text{ is unique minimizer of } J(w) \text{ over all } w \text{ in } X.}$$

the Rayleigh-Ritz Method
the old-fashioned way

a key property

Let's say we have two functions, $\hat{w}_1(x)$, $\hat{w}_2(x)$.

Now, since u minimizes $J(w)$



$$\underline{J(u) < J(\hat{w}^1) < J(\hat{w}^2)} \Rightarrow \underline{\| \hat{w}^1 - u \|_J < \| \hat{w}^2 - u \|_J}$$

How can we exploit?

$$\left(\text{Norm: } \|z\|_J^2 \equiv \int_0^1 \left(\frac{dz}{dx}\right)^2 + \mu z^2 \, dx .\right)$$

a trial space:

Introduce n functions (all in X),

$$\psi_1(x), \psi_2(x), \ldots, \psi_n(x) ;$$

note $\psi_1(0) = \psi_2(0) = \cdots = \psi_n(0) = 0$.

Then write

unknown coefficients

known functions

$$\hat{u}_{RR}(x) = \sum_{j=1}^{n} \alpha_j \, \psi_j(x) \qquad \approx u(x)$$

Rayleigh-Ritz approximation

observe that $\hat{u}_{RR}(x)$ is in X since any α_j

$$\hat{u}_{RR}(0) = \sum_{j=1}^{n} \alpha_j \, \psi_j(0) = 0 .$$

minimization:

Choose $\alpha = (\alpha_1 \; \alpha_2 \; \ldots \; \alpha_n)^T$ such that

$$\underline{J(\hat{u}_{RR})} = J\left(\sum_{j=1}^{n} \alpha_j \psi_j\right)$$

$$< \underline{J\left(\sum_{j=1}^{n} \beta_j \psi_j\right)}$$

$J(w)$ for any other linear combination of trial functions

for any $\beta = (\beta_1 \; \beta_2 \; \ldots \; \beta_n)^T \neq \alpha$.

$$\Rightarrow \Big\| \sum_{j=1}^{n} \alpha_j \psi_j - u \Big\|_J < \Big\| \sum_{j=1}^{n} \beta_j \psi_j - u \Big\|_J$$

"BEST" LINEAR COMBINATION

a simple example: $n = 2$

1/6

Choose

$$\psi_1(x) = x, \quad \psi_2(x) = x^2 ;$$

note $\psi_1(0) = \psi_2(0)$, as required.

Thus express any candidate (linear combination of ψ's) as

$$\sum_{j=1}^{2} \beta_j \psi_j(x) = \beta_1 x + \beta_2 x^2 ,$$

and the best linear combination as

$$\sum_{j=1}^{2} \alpha_j \psi_j(x) = \alpha_1 x + \alpha_2 x^2 .$$

Now, for any $\beta = (\beta_1\ \beta_2)^T$,

$$J\Big(\sum_{j=1}^{2} \beta_j \psi_j\Big) = J(\beta_1 \psi_1 + \beta_2 \psi_2)$$
$$= J(\beta_1 x + \beta_2 x^2) \equiv \mathcal{J}(\beta_1, \beta_2);$$

note J(a function) vs. $\mathcal{J}$(two scalars)

To minimize J, set $\qquad \alpha$ is the minimizer

$$\left.\begin{aligned} \frac{\partial \mathcal{J}}{\partial \beta_1}(\alpha_1, \alpha_2) &= 0 \\ \frac{\partial \mathcal{J}}{\partial \beta_2}(\alpha_1, \alpha_2) &= 0 \end{aligned}\right\} \Rightarrow \alpha_1, \alpha_2 \Rightarrow \hat{u}_{2R} = \alpha_1 x + \alpha_2 x^2$$

(Also confirm Hessian is SPD.)

First, find $\mathcal{J}(\beta_1, \beta_2)$:

$$\mathcal{J}(\beta_1, \beta_2) = J(\beta_1 x + \beta_2 x^2) \qquad f = -\mu$$

$$= \frac{1}{2}\int_0^1 \Big(\frac{d}{dx}(\beta_1 x + \beta_2 x^2)\Big)^2 + \mu(\beta_1 x + \beta_2 x^2)^2 + 2\mu(\beta_1 x + \beta_2 x^2)\, dx$$

$$= \frac{1}{2}\int_0^1 (\beta_1 + 2\beta_2 x)^2 + \mu(\beta_1 x + \beta_2 x^2)^2 + 2\mu(\beta_1 x + \beta_2 x^2)\, dx$$

$$= \frac{1}{2}\int_0^1 \beta_1^2 + 4\beta_1\beta_2 x + 4\beta_2^2 x^2 + \mu\beta_1^2 x^2 + 2\mu\beta_1\beta_2 x^3 + \mu\beta_2^2 x^4$$
$$+ 2\mu\beta_1 x + 2\mu\beta_2 x^2\, dx$$

$$= \boxed{\frac{1}{2}\Big(\beta_1^2 + 2\beta_1\beta_2 + \frac{4}{3}\beta_2^2 + \frac{\mu}{3}\beta_1^2 + \frac{\mu}{2}\beta_1\beta_2 + \frac{\mu}{5}\beta_2^2 + \mu\beta_1 + \frac{2\mu}{3}\beta_2\Big)}\ .$$

Then, differentiate:

$$\frac{\partial \mathcal{J}}{\partial \beta_1}(\beta_1, \beta_2) = \beta_1 + \beta_2 + \frac{\mu}{3}\beta_1 + \frac{\mu}{4}\beta_2 + \frac{\mu}{2}$$
$$= \underline{\Big(1 + \frac{\mu}{3}\Big)\beta_1 + \Big(1 + \frac{\mu}{4}\Big)\beta_2 + \frac{\mu}{2}}$$

and

$$\frac{\partial \mathcal{J}}{\partial \beta_2}(\beta_1, \beta_2) = \beta_1 + \frac{4}{3}\beta_2 + \frac{\mu}{4}\beta_1 + \frac{\mu}{5}\beta_2 + \frac{\mu}{3}$$
$$= \underline{\Big(1 + \frac{\mu}{4}\Big)\beta_1 + \Big(\frac{4}{3} + \frac{\mu}{5}\Big)\beta_2 + \frac{\mu}{3}}$$

Next, set derivatives to zero at $\beta = \alpha$:

$$\frac{\partial \mathcal{J}}{\partial \beta_1}(\beta_1 = \alpha_1, \beta_2 = \alpha_2) = 0$$
$$\Rightarrow \underline{\Big(1 + \frac{\mu}{3}\Big)\alpha_1 + \Big(1 + \frac{\mu}{4}\Big)\alpha_2 + \frac{\mu}{2} = 0}\ ;$$

$$\frac{\partial \mathcal{J}}{\partial \beta_2}(\beta_1 = \alpha_1, \beta_2 = \alpha_2) = 0$$
$$\Rightarrow \underline{\Big(1 + \frac{\mu}{4}\Big)\alpha_1 + \Big(\frac{4}{3} + \frac{\mu}{5}\Big)\alpha_2 + \frac{\mu}{3} = 0}\ .$$

(check Hessian)

Finally, organize as linear system:

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$$\boxed{\underbrace{\begin{pmatrix} 1+\frac{\mu}{3} & 1+\frac{\mu}{4} \\ 1+\frac{\mu}{4} & \frac{4}{3}+\frac{\mu}{5} \end{pmatrix}}_{\substack{A_{RR} \\ n=2 \times n=2}} \underbrace{\begin{pmatrix} \alpha_1 \\ \alpha_2 \end{pmatrix}}_{\substack{\alpha \\ n=2 \times 1}} = \underbrace{\begin{pmatrix} \frac{-\mu}{2} \\ \frac{-\mu}{3} \end{pmatrix}}_{\substack{F_{RR} \\ n=2 \times 1}} .}$$

$\Rightarrow \alpha$ DEMO

general n

Given $\psi_1(x), \dots, \psi_n(x)$, then

$$\hat{u}_{RR}(x) = \sum_{j=1}^{n} \alpha_j \psi_j(x)$$

for α solution of

$$\boxed{\underset{n\times n}{A_{RR}} \underset{n\times 1}{\alpha} = \underset{n\times 1}{F_{RR}}}$$

with

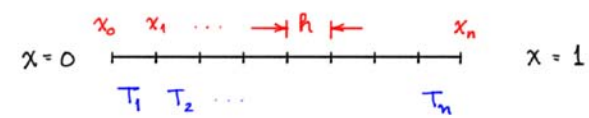
$$\begin{cases} A_{RR\,ij} = \int_0^1 \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \mu \psi_i \psi_j \, dx , \\ F_{RR\,i} = \int_0^1 \psi_i f \, dx \quad \left(= -\int_0^1 \psi_i \mu \, dx \right), \end{cases} \quad 1 \le i,j \le n .$$

the Rayleigh-Ritz Method
a "modern" set of ψ's

$\Downarrow$

the Finite Element Method (1-d)

a mesh



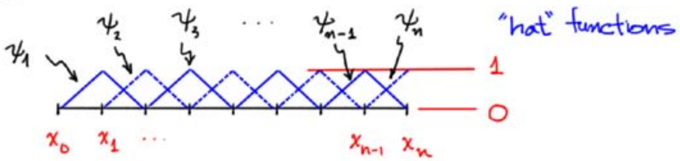
nodes (here equi-spaced): $h = 1/n$

$x_i = ih, \quad 0 \le i \le n$

elements:

$T_1 =]x_0, x_1[$, $T_2 =]x_1, x_2[$, … $T_n =]x_{n-1}, x_n[$

the ψ's



Note

(i) $\psi_i(0) = 0$, $1 \le i \le n$ $\quad (\Rightarrow \text{in } X)$

(ii) $\psi_i(x_j) = \delta_{ij}$, $1 \le i,j \le n$ $\quad$ nodal basis

(iii) $\psi_i(x)$ are piecewise linear

(iv) ψ_i "overlaps" only with ψ_{i-1}, ψ_i, and ψ_{i+1}

(Kronecker delta: $\delta_{ij} = 0$, $i \ne j$; $\delta_{ii} = 1$.)

RR approximation

$$\hat{u}_{RR}(x) \equiv u_h(x) = \sum_{j=1}^{n} \alpha_j \psi_j(x)$$

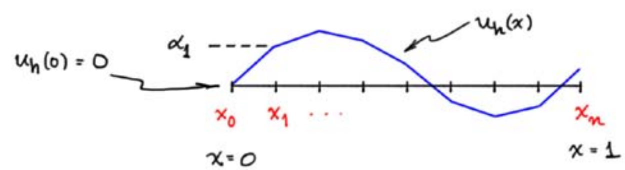
from minimization procedure

Note

(i) $u_h(x_i) = \sum_{j=1}^{n} \alpha_j \psi_j(x_i) = \alpha_i$

nodal value of u_h

(ii) $u_h(x)$ is piecewise linear



linear system (for $\alpha = (\alpha_1\ \alpha_2 \cdots \alpha_n)^T$)

$$\underbrace{A_h}_{\substack{A_{RR} \\ n\times n}} \underbrace{\alpha}_{n\times 1} = \underbrace{F_h}_{\substack{F_{RR} \\ n\times 1}} \quad \longleftarrow \text{RR minimization}$$

with

$$\begin{cases} A_{h\,ij} = \displaystyle\int_0^1 \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \mu\psi_i\psi_j\,dx \\ F_{h\,i} = \displaystyle\int_0^1 \psi_i f\,dx \quad \left(= -\int_0^1 \mu\psi_i\,dx\right) \end{cases} \quad 1 \le i,j \le n$$

sparsity

Since ψ_i overlaps only with ψ_{i-1}, ψ_i, ψ_{i+1},

$$A_{h\,ij} = \int_0^1 \frac{d\psi_i}{dx}\frac{d\psi_j}{dx} + \mu\psi_i\psi_j\,dx$$

$$= 0 \quad \text{unless } j = i-1, i, \text{ or } i+1$$

$$\Downarrow$$

$\underset{n\times n}{A_h}$ is sparse, and in fact tri-diagonal $\qquad$ (also SPD)

$$\boxed{\Rightarrow O(n) \text{ FLOPs to find } \alpha}$$

A_h, F_h in detail

direct stiffness assembly

$$A_h = \frac{1}{h}\begin{pmatrix} 2 & -1 & & & & \\ -1 & 2 & -1 & & 0 & \\ & -1 & 2 & -1 & & \\ & 0 & & \ddots & & \\ & & & & 2 & -1 \\ & & & & -1 & 1 \end{pmatrix}$$

$$\mu h \begin{pmatrix} 2/3 & 1/6 & & & & \\ 1/6 & 2/3 & 1/6 & & 0 & \\ & 1/6 & 2/3 & 1/6 & & \\ & 0 & & \ddots & & \\ & & & & 2/3 & 1/6 \\ & & & & 1/6 & 1/3 \end{pmatrix}$$

$$F_h = -h\left(\mu \ \mu \ \cdots \ \mu \ \tfrac{\mu}{2}\right)^T$$

Convergence

It can be shown that

$$\| u - u_h \|_J \le C h \qquad (C \text{ independent of } h);$$

note $\| \cdot \|_J$ measures function and derivative.

temperature heat flux

It can also be shown that for many "outputs" $\mathcal{O}$,

$$|\mathcal{O} - \mathcal{O}_h| \le C h^2$$

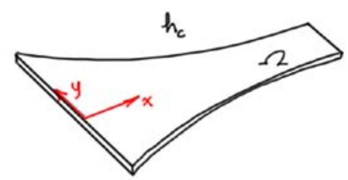
for example, $\mathcal{O} \equiv -du/dx$ (flux) [written properly].

DEMO

the Finite Element Method (2-d)

PDE BVP

a Partial Differential Equation



$$\begin{cases} -\left(\dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2}\right) + \mu u = f \quad \text{in } \Omega \\ \text{temperature or flux boundary conditions} \end{cases}$$

Γ_1 Γ_2

the Minimization Statement

The solution u to our PDE BVP

minimizes $J(w)$ over all w in X

for

$$J(w) \equiv \frac{1}{2}\int_\Omega \left(\frac{\partial w}{\partial x}\right)^2 + \left(\frac{\partial w}{\partial y}\right)^2 + \mu w^2 - 2fw \; dx\,dy$$

and

$$X \equiv \{ w \text{ suitably smooth}, \; w = 0 \text{ on } \Gamma_1 \} .$$

Rayleigh-Ritz Approximation

$$u_h(x,y) = \sum_{j=1}^{n} \alpha_j \psi_j(x,y)$$

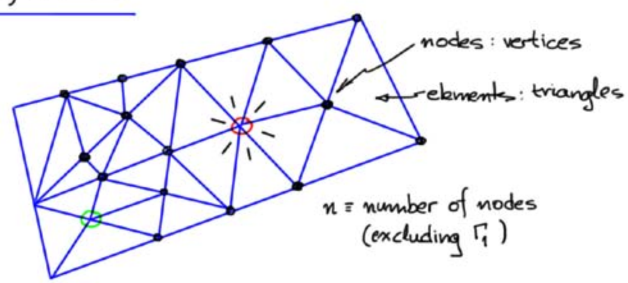
where

$$\underset{n\times n}{A_h}\, \underset{n\times 1}{\alpha} = \underset{n\times 1}{F_h}$$

and

$$\begin{cases} A_{h\,ij} = \int_\Omega \frac{\partial \psi_i}{\partial x}\frac{\partial \psi_j}{\partial x} + \frac{\partial \psi_i}{\partial y}\frac{\partial \psi_j}{\partial y} + \mu \psi_i \psi_j \; dx\,dy \\ F_{h\,i} = \int_\Omega \psi_i f \; dx\,dy \end{cases} \quad 1 \le i,j \le n .$$

a mesh, the ψ's



$\psi_{\circ}$: linear $(ax + by + c)$ in each participating element ; continuous

sparsity in A_h: no overlap between $\psi_{\circ}$ and $\psi_{\circ}$

DEMO : 3-d robot arm ; elasticity