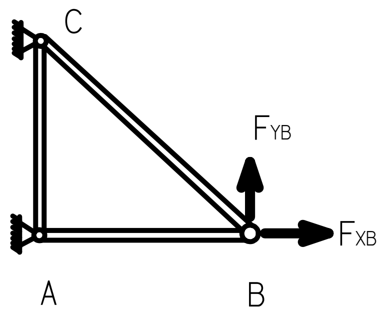
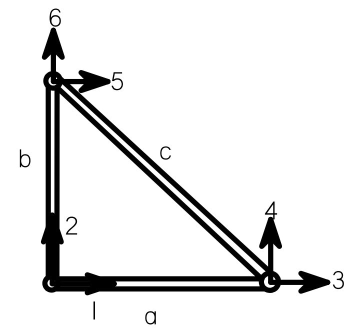
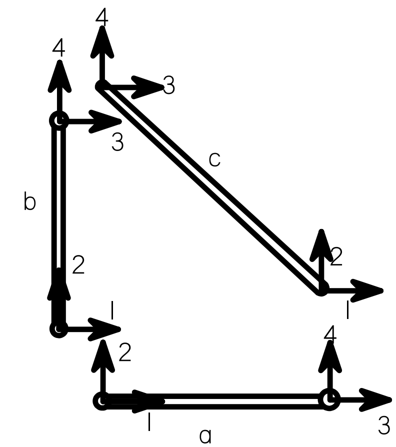
Matrix Analysis Example Hughes figure 5.12 page 191 ff

ORIGIN := 1







input data

n_elements := 3 n_nodes := 3 ie := 1 .. n_elements

f, δ element; F, Δ structure; m = element

n_free := 2 number of degrees of freedom per node

n_dof := n_nodes·n_free n_dof = 6 total number of degrees of freedom in structure

input for the class and text problem:

nod_el := 2 nodes per element in := 1 .. nod_el

$$elem := \begin{pmatrix} 1 & 2 \\ 1 & 3 \\ 2 & 3 \end{pmatrix}$$ nodal map of elements

$$XY := \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$ location of nodes

$$A := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} \qquad E := \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}$$

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element stiffness matrix: geometry

$$X_{ie,in} := XY_{elem_{ie,in},1} \qquad Y_{ie,in} := XY_{elem_{ie,in},2}$$

$$X = \begin{pmatrix} 0 & 1 \\ 0 & 0 \\ 1 & 0 \end{pmatrix} \qquad Y = \begin{pmatrix} 0 & 0 \\ 0 & 1 \\ 0 & 1 \end{pmatrix}$$

$$L_{ie} := \sqrt{\left(X_{ie,2} - X_{ie,1}\right)^2 + \left(Y_{ie,2} - Y_{ie,1}\right)^2} \qquad L = \begin{pmatrix} 1 \\ 1 \\ 1.414 \end{pmatrix}$$

$$angle_{ie} := \text{if}\left(\left|X_{ie,2} - X_{ie,1}\right| > 0, \text{atan}\left(\frac{Y_{ie,2} - Y_{ie,1}}{X_{ie,2} - X_{ie,1}}\right), \frac{\pi}{2}\right)$$

gets angle $-\pi/2$ < angle < $\pi/2$

$$\frac{angle}{deg} = \begin{pmatrix} 0 \\ 90 \\ -45 \end{pmatrix}$$

don't need angle now but will later for T

$$angle_{ie} := \text{if}\left(X_{ie,2} - X_{ie,1} < 0, angle_{ie} + \pi, angle_{ie}\right)$$

gets angle in appropriate quadrant

$$\frac{angle}{deg} = \begin{pmatrix} 0 \\ 90 \\ 135 \end{pmatrix}$$

element stiffness, element coordinates

$$ke_{ie} := \frac{A_{ie} \cdot E_{ie}}{L_{ie}} \cdot \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ke_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ke_2 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad ke_3 = \begin{pmatrix} 0.707 & 0 & -0.707 & 0 \\ 0 & 0 & 0 & 0 \\ -0.707 & 0 & 0.707 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

transformation matrix

$$\lambda_{ie} := \cos\left(angle_{ie}\right) \qquad \mu_{ie} := \sin\left(angle_{ie}\right) \qquad \lambda = \begin{pmatrix} 1 \\ 0 \\ -0.707 \end{pmatrix} \qquad \mu = \begin{pmatrix} 0 \\ 1 \\ 0.707 \end{pmatrix}$$

transform from structure to element; applies at each node of element.

$$T_{ie} := \begin{bmatrix} \lambda_{ie} & \mu_{ie} & 0 & 0 \\ -\mu_{ie} & \lambda_{ie} & 0 & 0 \\ 0 & 0 & \lambda_{ie} & \mu_{ie} \\ 0 & 0 & (-\mu)_{ie} & \lambda_{ie} \end{bmatrix} \quad T_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad T_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad T_3 = \begin{pmatrix} -0.707 & 0.707 & 0 & 0 \\ -0.707 & -0.707 & 0 & 0 \\ 0 & 0 & -0.707 & 0.707 \\ 0 & 0 & -0.707 & -0.707 \end{pmatrix}$$

element stiffness, structure coordinates

$$Ke_{ie} := T_{ie}^{T} \cdot ke_{ie} \cdot T_{ie} \quad Ke_1 = \begin{pmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad Ke_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 1 \end{pmatrix} \quad Ke_3 = \begin{pmatrix} 0.354 & -0.354 & -0.354 & 0.354 \\ -0.354 & 0.354 & 0.354 & -0.354 \\ -0.354 & 0.354 & 0.354 & -0.354 \\ 0.354 & -0.354 & -0.354 & 0.354 \end{pmatrix}$$

assemble structure stiffness matrix structure coordinates

now we have to deal with total structure model:

$$F = \begin{pmatrix} F_1 \\ F_2 \\ F_3 \\ F_4 \\ F_5 \\ F_6 \end{pmatrix} \qquad \Delta = \begin{pmatrix} \Delta_1 \\ \Delta_2 \\ \Delta_3 \\ \Delta_4 \\ \Delta_5 \\ \Delta_6 \end{pmatrix}$$

and $F = K \cdot \Delta$ superposing respective element contributions

convert node number to numbered degree of freedom

$j := 1 .. n\_free \qquad k := 0 .. n\_free - 1 \qquad top_{ie, 2 \cdot j - k} := n\_free\, elem_{ie, j} - k \qquad top = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 5 & 6 \\ 3 & 4 & 5 & 6 \end{pmatrix}$

$i := 1 .. nod\_el\, n\_free \qquad j := 1 .. nod\_el\, n\_free$

$K_{n\_dof, n\_dof} := 0$

$K_{top_{ie,i}, top_{ie,j}} := K_{top_{ie,i}, top_{ie,j}} + \left(Ke_{ie}\right)_{i,j}$

$$K = \begin{pmatrix} 1 & 0 & -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ -1 & 0 & 1.354 & -0.354 & -0.354 & 0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & 0 & -0.354 & 0.354 & 0.354 & -0.354 \\ 0 & -1 & 0.354 & -0.354 & -0.354 & 1.354 \end{pmatrix}$$

set up forces, lhs of F = K*Δ

$ii := 1 .. n\_dof \qquad F_{ii} := 0 \qquad \begin{pmatrix} F_3 \\ F_4 \end{pmatrix} := \begin{pmatrix} 3 \\ 4 \end{pmatrix} \qquad F \rightarrow \begin{pmatrix} 0 \\ 0 \\ 3 \\ 4 \\ 0 \\ 0 \end{pmatrix}$

apply boundary conditions

only degrees of freedom 3 and 4 are unconstrained therefore the reduced equations become

$F_{red} := \text{submatrix}(F,3,4,1,1)$ $\quad F_{red} \rightarrow \begin{pmatrix} 3 \\ 4 \end{pmatrix}$ $\quad K_{red} := \text{submatrix}(K,3,4,3,4)$ $\quad K_{red} = \begin{pmatrix} 1.354 & -0.354 \\ -0.354 & 0.354 \end{pmatrix}$

solve for Δ and F

$\Delta_{ii} := 0$ and we can solve for Δ3 and Δ4 $\quad \begin{pmatrix} \Delta_3 \\ \Delta_4 \end{pmatrix} := K_{red}^{-1} \cdot F_{red}$ $\quad F := K \cdot \Delta$ $\quad F = \begin{pmatrix} -7 \\ 0 \\ 3 \\ 4 \\ 4 \\ -4 \end{pmatrix}$ $\quad \Delta = \begin{pmatrix} 0 \\ 0 \\ 7 \\ 18.314 \\ 0 \\ 0 \end{pmatrix}$

reverse to calculate element properties

and then the element forces are calculated from the relationships that we began with:

first get Delta (structure coordinates) of each element

$\Delta e_{ie,i} := \Delta_{top_{ie,i}}$ $\quad \Delta e = \begin{pmatrix} 0 & 0 & 7 & 18.314 \\ 0 & 0 & 0 & 0 \\ 7 & 18.314 & 0 & 0 \end{pmatrix}$ $\quad \Delta e^T = \begin{pmatrix} 0 & 0 & 7 \\ 0 & 0 & 18.314 \\ 7 & 0 & 0 \\ 18.314 & 0 & 0 \end{pmatrix}$ $\quad \delta_{ie} := T_{ie} \cdot \left(\Delta e^T\right)^{\langle ie \rangle}$ $\quad \delta_1 = \begin{pmatrix} 0 \\ 0 \\ 7 \\ 18.314 \end{pmatrix}$ $\quad \delta_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad \delta_3 = \begin{pmatrix} 8 \\ -17.899 \\ 0 \\ 0 \end{pmatrix}$

$f_{ie} := ke_{ie} \cdot \delta_{ie}$ $\quad f_1 = \begin{pmatrix} -7 \\ 0 \\ 7 \\ 0 \end{pmatrix}$ $\quad f_2 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$ $\quad f_3 = \begin{pmatrix} 5.657 \\ 0 \\ -5.657 \\ 0 \end{pmatrix}$

apply stress matrix

$Se_{ie} := \frac{E_{ie}}{L_{ie}} \cdot (-1 \ \ 0 \ \ 1 \ \ 0)$ $\quad Se_1 = (-1 \ \ 0 \ \ 1 \ \ 0)$ $\quad Se_2 = (-1 \ \ 0 \ \ 1 \ \ 0)$ $\quad Se_3 = (-0.707 \ \ 0 \ \ 0.707 \ \ 0)$

$\sigma_{ie} := Se_{ie} \cdot \delta_{ie}$ $\quad \sigma = \begin{pmatrix} 7 \\ 0 \\ -5.657 \end{pmatrix}$