Lecture 6 - 2003

Torsion Properties for Line Segments and Computational Scheme for Piecewise Straight Section Calculations

this consists of four parts (and how we will treat each)
A - derivation of geometric algorithms for section properties (cover quickly for sense of approach)
B - derivation of first moment approach (for info - not covered)
C - computational routine resulting from A (demo a few examples - routine available in lab)
D - computational routine resulting from B (routine available in lab)

sourced from section 6.1 to 6.3 of Kollbrunner, Curt Friedrich, Torsion in structures; an engineering approach TA417.7.T6.K811 1966, and geometry.

starting point: a line defined by two points, x1,y1 and x0,y0

this assumes X.cg and Ycg known

$$\frac{y - y_0}{x - x_0} = \frac{y_1 - y_0}{x_1 - x_0}$$

line passing through two points

$$y = \frac{y_1 - y_0}{x_1 - x_0}\cdot x + y_0 - x_0\frac{(y_1 - y_0)}{(x_1 - x_0)} \text{ or ... } \quad x = \frac{x_1 - x_0}{y_1 - y_0}\cdot y + x_0 - y_0\cdot\frac{x_1 - x_0}{y_1 - y_0}$$

consider calculation of increment of moment of inertia (relative to centroid)

$$\int_0^b y^2\cdot t\, ds \qquad ds = \frac{dx}{\cos(\alpha)} \qquad \Delta s = \text{length} = \frac{\Delta x}{\cos(\alpha)} \qquad \int_0^b y^2\cdot t\, ds = \frac{t}{\cos(\alpha)}\cdot\int_{x_0}^{x_1} y^2\, dx$$

$$\int_{x_0}^{x_1}\left[\frac{y_1 - y_0}{x_1 - x_0}\cdot x + y_0 - x_0\cdot\left(\frac{y_1 - y_0}{x_1 - x_0}\right)\right]^2 dx \;\Bigg|\begin{matrix}\text{simplify}\\ \text{factor}\end{matrix} \rightarrow \frac{1}{3}\cdot\left(y_0^{\,2} + y_1\cdot y_0 + y_1^{\,2}\right)\cdot(x_1 - x_0)$$

$$\int_0^b y^2\cdot t\, ds = \frac{t}{\cos(\alpha)}\cdot\int_{x_0}^{x_1} y^2\, dx = \frac{t}{\cos(\alpha)}\cdot\left[\frac{(x_1 - x_0)}{3}\cdot\left[(y_1)^2 + y_0\cdot y_1 + (y_0)^2\right]\right]$$

$$I_x = \frac{t\cdot(s_1 - s_0)}{3}\cdot\left[(y_1)^2 + y_0\cdot y_1 + (y_0)^2\right]$$

similarly (by the symmetry of the expression for the line above):

$$I_y = \frac{t\cdot(s_1 - s_0)}{3}\cdot\left[(x_1)^2 + x_0\cdot x_1 + (x_0)^2\right]$$

cross moment of inertia

$$I_{xy} = \int_0^b x\cdot y\cdot t\, ds = \frac{t}{\cos(\alpha)}\cdot\int_{x_0}^{x_1} x\cdot y\, dx = \frac{t}{\sin(\alpha)}\cdot\int_{y_0}^{y_1} x\cdot y\, dy$$

$$\int_{x_0}^{x_1} x\cdot y\,dx = \int_{x_0}^{x_1}\left[\frac{y_1-y_0}{x_1-x_0}\cdot x + y_0 - x_0\cdot\left(\frac{y_1-y_0}{x_1-x_0}\right)\right]\cdot x\,dx$$

$$\int_{x_0}^{x_1}\left[\frac{y_1-y_0}{x_1-x_0}\cdot x + y_0 - x_0\cdot\left(\frac{y_1-y_0}{x_1-x_0}\right)\right]\cdot x\,dx \;\Big|\begin{array}{l}\text{simplify}\\ \text{factor}\end{array} \rightarrow \frac{1}{6}\cdot(x_1-x_0)\cdot(2\cdot x_1\cdot y_1 + y_0\cdot x_1 + x_0\cdot y_1 + 2\cdot y_0\cdot x_0)$$

$$I_{xy} = \frac{t}{\cos(\alpha)}\cdot\int_{x_0}^{x_1} x\cdot y\,dx = \frac{t}{6}\cdot\left(\frac{x_1-x_0}{\cos(\alpha)}\right)\cdot\left[2\cdot(x_1\cdot y_1 + x_0\cdot y_0) + x_0\cdot y_1 + x_1\cdot y_0\right] = \frac{t\cdot(s_1-s_0)}{6}\cdot\left[\left[\begin{array}{l}2\cdot(x_1\cdot y_1 + x_0\cdot y_0)\ \ldots\\ +x_0\cdot y_1 + x_1\cdot y_0\end{array}\right]\right]$$

we could calculate Iyx using the same relationship but we know it is = Ixy $\qquad I_{yx} = I_{xy}$

to evaluate the warping relationships: start with line passing through two points and obtain normal form of line

$$\frac{y-y_0}{x-x_0} = \frac{y_1-y_0}{x_1-x_0} \qquad \text{or ...} \qquad y = \frac{y_1-y_0}{x_1-x_0}\cdot x + y_0 - x_0\frac{(y_1-y_0)}{(x_1-x_0)}$$

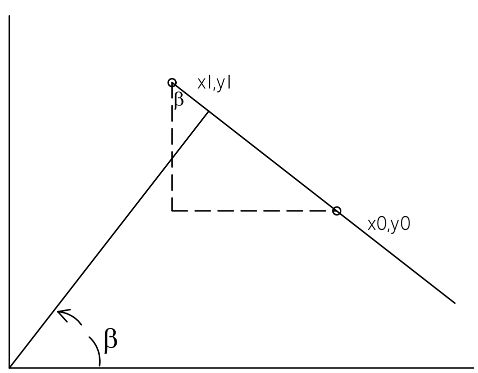
multiply by (x1 - x0) $\qquad y\cdot(x_1-x_0) = (y_1-y_0)\cdot x + y_0\cdot(x_1-x_0) - x_0(y_1-y_0)$

rearrange => $\qquad -(y_1-y_0)\cdot x + (x_1-x_0)\cdot y + x_0\cdot(y_1-y_0) - y_0\cdot(x_1-x_0) = 0$

$$A = -(y_1-y_0) \qquad B = x_1-x_0 \qquad C = x_0\cdot(y_1-y_0) - y_0\cdot(x_1-x_0)$$

general form: $\qquad A\cdot x + B\cdot y + C = 0$

to reduce to normal form x*cos(β)+y*sin(β)=p divide by $\sqrt{(y_1-y_0)^2 + (x_1-x_0)^2}$ where sign is opposite of C. $C \neq 0$ and β is angle between the x axis and the NORMAL to line.



$$denom = \sqrt{(y_1-y_0)^2 + (x_1-x_0)^2} \qquad denom = -denom\cdot sign(C)$$

for the geometry shown: x0 > x1, y1 > y0

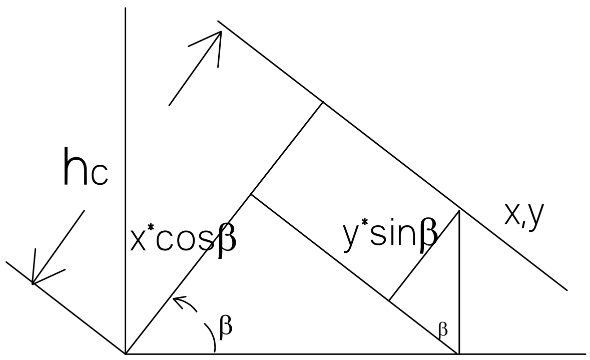
$$\sin(\beta) = \frac{-(x_1-x_0)}{denom} \qquad \cos(\beta) = \frac{y_1-y_0}{denom}$$

$A\cdot x + B\cdot y + C = 0$ becomes $\quad -A\cdot x - B\cdot y = C$

$$\frac{y_1-y_0}{denom}\cdot x + \frac{(x_1-x_0)}{denom}\cdot y = \frac{C}{denom} \qquad \rho_c = \frac{C}{denom}$$

$$x\cdot\cos(\beta) + y\cdot\sin(\beta) = \rho_c$$

we could also have determined this direct from the geometry
x, y is a point on a line a distance hc from the centroid:



$$x\cdot\cos(\beta) + y\cdot\sin(\beta) = h_c$$

$d\omega_c = d\Omega_c = \rho_c\cdot ds$ definition of $\omega_c = \int h_c\, ds$

for a straight line segment $\rho_c = \text{constant}$ $\Delta\omega_c = \rho_c\cdot L$ and is linear along line

$\rho_c = p$ from normal form of line

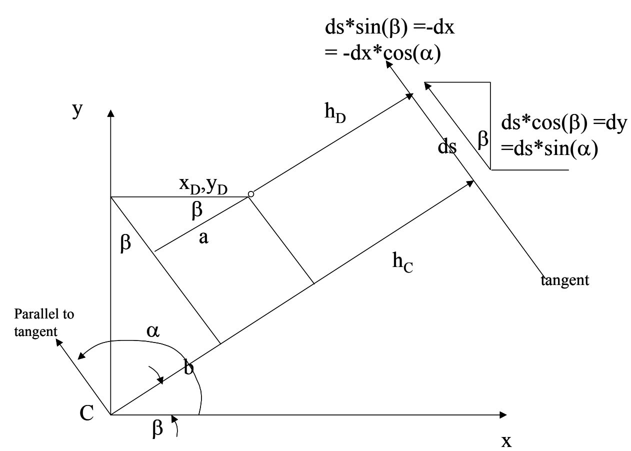
ρ_c is positive if centroid is to the left when viewing the element from i to j (0 to 1) along the tangent line

alternative form of line (cos(α), sin(α) and p defined in terms of x1,y1 x0,y0 above in this form p is the distance from origin to line and β is angle NORMAL to line makes with x axis

$$x\cdot\cos(\beta) + y\cdot\sin(\beta) = \rho_c = h_c$$

the increase in ωc due to this line segment is then

$$\Delta\omega_c = \rho_c\cdot L = \int_{s_0}^{s_1} h_c\, ds = \int_{s_0}^{s_1} x\cdot\cos(\beta)\, ds + \int_{s_0}^{s_1} y\cdot\sin(\beta)\, ds = \int_{y_0}^{y_1} x\, dy - \int_{x_0}^{x_1} y\, dx$$



"it can be shown"

$$\int_{s_0}^{s_1} h_c\,ds = \int_{y_0}^{y_1} x\,dy - \int_{x_0}^{x_1} y\,dx = \left(\frac{x_1 + x_0}{2}\right)\cdot\left(y_1 - y_0\right) - \frac{\left(y_1 + y_0\right)}{2}\cdot\left(x_1 - x_0\right)$$

$$\Delta\omega_c = \frac{x_1 + x_0}{2}\cdot\left(y_1 - y_0\right) - \frac{y_1 + y_0}{2}\cdot\left(x_1 - x_0\right)$$

$$\Delta\omega_c = xm\cdot(\Delta y) - ym\cdot\Delta x$$

xm = mid-point
Δx = x1 - x0
Δy = y1 - y0

$$I_{x\omega c} = \int_0^b \omega_c\cdot y\,ds \qquad I_{y\omega c} = \int_0^b \omega_c\cdot x\,ds$$

ω_c is linear with s for a line h_c is constant => $\omega_c = \int h_c\,ds = h_c\cdot\int 1\,ds = h_c\cdot s$

initial value is ω0 and end value ω1
being linear with s also implies linear with x and y i.e.
with x

$$\omega_c(s) = \omega_{c0} + \left(\omega_{c1} - \omega_{c0}\right)\frac{\left(x - x_0\right)}{x_1 - x_0} = \left(\frac{\omega_{c1} - \omega_{c0}}{x_1 - x_0}\right)x + \omega_{c0} - \omega_{c0}\frac{\left(x - x_0\right)}{x_1 - x_0}$$

which is exactly like
with y substituted for ω

$$y = \frac{y_1 - y_0}{x_1 - x_0}\cdot x + y_0 - x_0\frac{\left(y_1 - y_0\right)}{\left(x_1 - x_0\right)}$$

so ...... just as $I_{xy} = \int_0^b x\cdot y\,dA$ $\qquad I_{xy} = \frac{t\cdot\left(s_1 - s_0\right)}{6}\cdot\left[2\cdot\left(x_1\cdot y_1 + x_0\cdot y_0\right) + x_0\cdot y_1 + x_1\cdot y_0\right]$

$$I_{y\omega c} = \int_0^b \omega_c\cdot x\,ds = \int_0^b \left[\left(\frac{\omega_{c1} - \omega_{c0}}{x_1 - x_0}\right)\cdot x + \omega_{c0} - \omega_{c0}\cdot\frac{\left(x - x_0\right)}{x_1 - x_0}\right]\cdot x\,ds$$

$$I_{y\omega c} = \frac{t\cdot\left(s_1 - s_0\right)}{6}\cdot\left[2\cdot\left(x_1\cdot\omega c_1 + x_0\cdot\omega c_0\right) + x_0\cdot\omega c_1 + x_1\cdot\omega c_0\right]$$

and ..... $I_{x\omega c} = \frac{t\cdot\left(s_1 - s_0\right)}{6}\cdot\left[2\cdot\left(y_1\cdot\omega_{c1} + y_0\cdot\omega c_0\right) + y_0\cdot\omega c_1 + y_1\cdot\omega c_0\right]$

now we can locate the shear center: (assume for the time being that these values are the results for a more complete section - we'll tie this together later)

from previous lecture

$$y_D := \frac{\left(I_{y\omega c} \cdot I_z - I_{yz} \cdot I_{z\omega c}\right)}{\left(I_y \cdot I_z - I_{yz}^2\right)} \qquad z_D := \frac{\left(-I_{z\omega c} \cdot I_y + I_{yz} \cdot I_{y\omega c}\right)}{\left(I_y \cdot I_z - I_{yz}^2\right)}$$

the coordinate system is changed from y,z to x,y changing y to x (first) and then z to y

$$x_D := \frac{\left(I_{x\omega c} \cdot I_y - I_{xy} \cdot I_{y\omega c}\right)}{\left(I_x \cdot I_y - I_{xy}^2\right)} \qquad y_D := \frac{\left(-I_{y\omega c} \cdot I_x + I_{xy} \cdot I_{x\omega c}\right)}{\left(I_x \cdot I_y - I_{xy}^2\right)}$$

now we can calculate ωD by first calculating Ω (this is warping referenced to shear center with an arbitrary coordinate system.

β is the same as our angle α $\quad h_D \cdot ds = h_C \cdot ds - x_D \cdot \sin(\alpha) \cdot ds + y_D \cdot \cos(\alpha) \cdot ds$

$$d\Omega_D = d\omega_D = h_D \cdot ds \qquad \Rightarrow \qquad \Omega_D(s) = \int_0^s h_D \, ds = \int_0^s h_C - x_D \cdot \sin(\alpha) + y_D \cdot \cos(\alpha) \, ds$$

$$\Omega_D(s) = \int_0^s h_C \, ds - \int_0^s x_D \cdot \sin(\alpha) \, ds - \int_0^s y_D \cdot \cos(\alpha) \, ds = \omega c(s) - x_D \cdot \int_{y_0}^{y} 1 \, dy + y_D \cdot \int_{x_0}^{x} 1 \, dx$$ where α constant

as ... $\qquad ds \cdot \cos(\alpha) = dx \qquad ds \cdot \sin(\alpha) = dy$

$$\Delta\Omega_D(s) = \Delta\omega c - x_D \cdot \left(y_1 - y_0\right) + y_D \cdot \left(x_1 - x_0\right)$$

if we set $\Omega_{D0} = 0$ at the start of a line segment, then $\Omega_{D1} = \Omega_{D0} + \Delta\Omega_D$

we find the centroid as we would X and Y (it's linear therefore cg of each segment is (Ω1+Ω0)/2). Then the normalized ΩD i.e.ωD is. ΩD-ΩDo and the moments are calculated as above

calculate "centroid" of warping wrt shear center:

$$\Delta Q_{\Omega D_i} = \frac{a_i}{2} \cdot \left(\Omega_{D_i} + \Omega_{D_{i+1}}\right) \qquad \Omega_{Dcg} = \frac{\sum \Delta Q_{\Omega D}}{A} \qquad \omega_{D_j} = \Omega_{D_j} - \Omega_{Dcg}$$

$$I_{\omega} = \frac{t\cdot(s_1 - s_0)}{3}\cdot\left[(\omega D_1)^2 + \omega D_0\cdot\omega D_1 + (\omega D_0)^2\right]$$

$$I_{y\omega D} = \frac{t\cdot(s_1 - s_0)}{6}\cdot\left[2\cdot(x_1\cdot\omega D_1 + x_0\cdot\omega D_0) + x_0\cdot\omega D_1 + x_1\cdot\omega D_0\right]$$

$$I_{x\omega D} = \frac{t\cdot(s_1 - s_0)}{6}\cdot\left[2\cdot(y_1\cdot\omega_{D1} + y_0\cdot\omega D_0) + y_0\cdot\omega D_1 + y_1\cdot\omega D_0\right]$$

these should be thought of in terms of contributions from the segment as we'll see in the overall scheme

the total should = 0

why?

$$\int_0^b \sigma\cdot x\,dA = -E\cdot\phi''\cdot\int_0^b \omega\cdot x\,dA = \int_0^b \omega\cdot x\,dA = 0 \qquad \int_0^b \sigma\cdot y\,dA = -E\cdot\phi''\cdot\int_0^b \omega\cdot y\,dA = \int_0^b \omega\cdot y\,dA = 0$$

as bending moments = 0

first moment approach
assume Xcg and Ycg known

$$Q_x = \int_{s_0}^{s_1} y\,dA$$

$$\int_{s_0}^{s_1} y\cdot t\,ds \qquad ds = \frac{dx}{\cos(\alpha)} \qquad \Delta s = \text{length} = \frac{\Delta x}{\cos(\alpha)} \qquad \int_0^b y\cdot t\,ds = \frac{t}{\cos(\alpha)}\cdot\int_{x_0}^{x_1} y\,dx$$

$$\int_{x_0}^{x_1}\left[\frac{y_1 - y_0}{x_1 - x_0}\cdot x + y_0 - x_0\cdot\left(\frac{y_1 - y_0}{x_1 - x_0}\right)\right]dx \;\Big|\begin{matrix}\text{simplify}\\ \text{factor}\end{matrix} \rightarrow \frac{1}{2}\cdot(x_1 - x_0)\cdot(y_1 + y_0) \quad \Rightarrow$$

$$\int_{s_0}^{s_1} y\cdot t\,ds = \frac{t}{\cos(\alpha)}\cdot\left[(x_1 - x_0)\cdot\left(\frac{y_1 + y_0}{2}\right)\right] = t\cdot(s_1 - s_0)\cdot\frac{y_1 + y_0}{2} = a_i\cdot\left(\frac{y_1 + y_0}{2}\right)$$

this should have been obvious as cg is mid point and moment of area is y_{cg} * area

$$ym_i = \frac{y_{i+1} + y_i}{2} \qquad xm_i = \frac{x_{i+1} + x_i}{2}$$

now for the moments of inertia:
we saw that:

$$I_x = \int_0^b y^2\cdot t\,ds = -\int_{y_0}^{y_1} Q_x\cdot t\,dy$$

now this presents a small problem:
Q is linear only where x or y is constant otherwise it's parabolic
this can be handled easily if we calculate the values at the midpoints and use Simpson's rule for integration: it is exact for a parabolic variation (linear is a subset order = 1) we will get 2*n_elements + 1 values
this time we'll keep a running total

increase is calculated using the approach above over each half (hence 1/2 of area and 1/2 of endpoints) of the segment:

$$\Delta Q_{x_{2\cdot i}} := \frac{a_i}{4}\cdot\left(y_i + ym_i\right) \quad \Delta Q_{x_{2\cdot i+1}} := \frac{a_i}{4}\cdot\left(ym_i + y_{i+1}\right) \qquad \Delta Q_{y_{2\cdot i}} := \frac{a_i}{4}\cdot\left(x_i + xm_i\right) \quad \Delta Q_{y_{2\cdot i+1}} := \frac{a_i}{4}\cdot\left(xm_i + x_{i+1}\right)$$

$k := 1 .. 2\cdot n\_elements$

$$Q_{x_0} := 0 \qquad Q_{x_k} = Q_{x_{k-1}} + \Delta Q_{x_{k-1}} \qquad Q_{y_0} := 0 \qquad Q_{y_k} = Q_{y_{k-1}} + \Delta Q_{y_{k-1}}$$

integration is over entire element using midpoint and endpoint values => n_element values

$$\Delta I_x = \int_{y_0}^{y_1} y^2\cdot t\,dy = -\int_{y_0}^{y_1} Q_x\cdot t\,dy = -t\,\frac{\left(y_1 - y_0\right)}{6}\cdot\left[Q_{x_{2\cdot i}} + 4\cdot Q_{x_{2\cdot i+1}} + Q_{x_{2\cdot(i+1)}}\right]$$

Simpson's rule

since this result will be useful later on we'll put it aside:

similarly for Iyy

$$Q_{x\_bar_i} = Q_{x_{2\cdot i}} + 4\cdot Q_{x_{2\cdot i+1}} + Q_{x_{2\cdot(i+1)}} \qquad Q_{y\_bar_i} = Q_{y_{2\cdot i}} + 4\cdot Q_{y_{2\cdot i+1}} + Q_{y_{2\cdot(i+1)}}$$

$$I_{xx} = \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q\_bar_{x_i}\cdot\Delta y_i \qquad I_{yy} = \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q\_bar_{y_i}\cdot\Delta x_i$$

the cross moments of inertia are:

$$I_{yz} = -\int_0^b Q_y\,dz = -\int_0^b Q_z\,dy$$

$$I_{xy} = \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q\_bar_{x_i}\cdot\Delta x_i \qquad I_{yx} = \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q\_bar_{y_i}\cdot\Delta y_i$$

derived above:

$$\Delta\omega_c = \frac{x_1 + x_0}{2}\cdot\left(y_1 - y_0\right) - \frac{y_1 + y_0}{2}\cdot\left(x_1 - x_0\right)$$

$$\Delta\omega_c = xm\cdot\left(\Delta y\right) - ym\cdot\Delta x$$

and in lecture 6

$$I_{y\omega} = -\int_0^b Q_y\,d\omega = -\int_0^b Q_\omega\,dx$$

$$I_{x\omega} = -\int_0^b Q_x\,d\omega = -\int_0^b Q_\omega\,dy$$

$$\Delta\omega_{c_i} = xm_i \cdot \Delta y_i - ym_i \cdot \Delta x_i \qquad I_{x\omega c} = \frac{-1}{6} \cdot \sum_{i=0}^{n\_elements-1} Q_{x\_bar_i} \cdot \Delta\omega_{c_i} \qquad I_{y\omega c} = \frac{-1}{6} \cdot \sum_{i=0}^{n\_elements-1} Q_{y\_bar_i} \cdot \Delta\omega_{c_i}$$

as above:

$$x_D := \frac{\left(I_{x\omega c} \cdot I_y - I_{xy} \cdot I_{y\omega c}\right)}{\left(I_x \cdot I_y - I_{xy}^2\right)} \qquad y_D := \frac{\left(-I_{y\omega c} \cdot I_x + I_{xy} \cdot I_{x\omega c}\right)}{\left(I_x \cdot I_y - I_{xy}^2\right)}$$

now we can calculate the warping parameters: as above: calculate ΩD and centroid

$$\Delta\Omega_D(s) = \Delta\omega c - x_D \cdot \left(y_1 - y_0\right) + y_D \cdot \left(x_1 - x_0\right)$$

if we set $\Omega_{D0} = 0$ at the start of a line segment, then $\Omega_{D1} = \Omega_{D0} + \Delta\Omega_D$

we find the centroid as we would X and Y (it's linear therefore cg of each segment is (Ω1+Ω0)/2). Then the normalized ΩD i.e.ωD is. ΩD-ΩDo and the moments are calculated as above

calculate "centroid" of warping wrt shear center:

$$\Delta Q_{\Omega D_i} = \frac{a_i}{2} \cdot \left(\Omega_{D_i} + \Omega_{D_{i+1}}\right) \qquad \Omega_{Dcg} = \frac{\sum \Delta Q_{\Omega D}}{A} \qquad \omega_{D_j} = \Omega_{D_j} - \Omega_{Dcg}$$

instead of direct integration based on the linear relationship as above we calculate the value at the mid-points and the Q

$$\omega m_{D_i} = \frac{\omega_{D_i} + \omega_{D_{i+1}}}{2}$$

now for the moments of inertia:
we saw that above (this was copied an x and y changed to ω:

$$I_\omega = \int_0^b \omega^2 \cdot t\, ds = -\int_{\omega_0}^{\omega_1} Q_\omega \cdot t\, d\omega$$

increase is calculated using the approach above over each half (hence 1/2 of area and 1/2 of endpoints) of the segment:

$$\Delta Q_{\omega_{2 \cdot i}} := \frac{a_i}{4} \cdot \left(\omega_i + \omega m_i\right) \Delta Q_{\omega_{2 \cdot i+1}} := \frac{a_i}{4} \cdot \left(\omega m_i + \omega_{i+1}\right)$$

$k := 1 .. 2 \cdot n\_elements$

$Q_{\omega_0} := 0$ $\qquad Q_{\omega_k} = Q_{\omega_{k-1}} + \Delta Q_{\omega_{k-1}}$

integration is over entire element using midpoint and endpoint values => n_element values

$$\Delta I_{\omega} = \int_{\omega_0}^{\omega_1} \omega^2 \cdot t \, d\omega = -\int_{\omega_0}^{\omega_1} Q_{\omega} \cdot t \, d\omega = -t \frac{\left(\omega_1 - \omega_0\right)}{6} \cdot \left[Q_{\omega_{2 \cdot i}} + 4 \cdot Q_{\omega_{2 \cdot i+1}} + Q_{\omega_{2 \cdot (i+1)}}\right]$$ Simpson's rule

since this result will be useful later on we'll put it aside:

$$Q_{\omega\_bar_i} = Q_{\omega_{2 \cdot i}} + 4 \cdot Q_{\omega_{2 \cdot i+1}} + Q_{\omega_{2 \cdot (i+1)}}$$

$$I_{\omega\omega} = \frac{-1}{6} \cdot \sum_{i=0}^{n\_elements-1} Q_{\omega\_bar_i} \cdot \Delta\omega_i$$

the cross moments are:

$$I_{x\omega} = \frac{-1}{6} \cdot \sum_{i=0}^{n\_elements-1} Q_{x\_bar_i} \cdot \Delta\omega_i \qquad I_{y\omega} = \frac{-1}{6} \cdot \sum_{i=0}^{n\_elements-1} Q_{y\_bar_i} \cdot \Delta\omega_i$$

Computational Scheme for Cross-Sectional Quantities

X, Y, 0 ... n_elements as get extra when start with 0
A 0 .... nelements -1

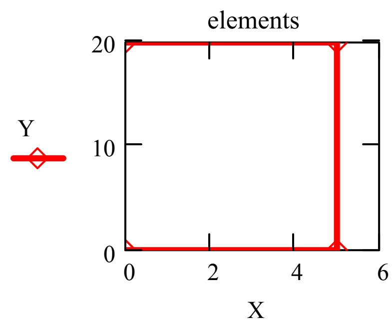
$X := \text{input}$ $\quad Y := \text{input}$ $\quad n\_elements := \text{input}$ $\quad a := \text{input}$ or ... $t := \text{input}$

$n\_elements := 3$

$i := 0 .. n\_elements - 1$ $\quad a_{n\_elements} := 0$ $\quad a_{n\_elements} := 0 \;\; t_{n\_elements} := 0$

$j := 0 .. n\_elements$ $\quad t_{n\_elements} := 0$

$$X := \begin{pmatrix} 0 \\ 5 \\ 5 \\ 0 \end{pmatrix} \qquad Y := \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix} \qquad t := \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$



we will use these later

$\Delta X_i := X_{i+1} - X_i$ $\quad \Delta x_i := \Delta X_i$ $\quad \Delta Y_i := Y_{i+1} - Y_i$ $\quad \Delta y_i := \Delta Y_i$

calculate area if necessary

$$a_i := \text{if}\left[a_i = 0, t_i \cdot \sqrt{(\Delta X_i)^2 + (\Delta Y_i)^2}, a_i\right] \qquad A := \sum a \qquad A = 15$$

calculate centroid in X and Y coordinate system and coordinates in centroidal system:

$$\Delta Q_{Y_i} := \frac{a_i}{2} \cdot (X_i + X_{i+1}) \qquad X_{cg} := \frac{\sum \Delta Q_Y}{A} \qquad X_{cg} = 4.167 \qquad x_j := X_j - X_{cg}$$

$$\Delta Q_{X_i} := \frac{a_i}{2} \cdot (Y_i + Y_{i+1}) \qquad Y_{cg} := \frac{\sum \Delta Q_X}{A} \qquad Y_{cg} = 10 \qquad y_j := Y_j - Y_{cg}$$

calculate moments of inertia

these are contributions from segment i = 0, t*(s1 - s0) = area of segment ai

$$I_x = \frac{t\cdot(s_1 - s_0)}{3}\cdot\left[\left(y_1\right)^2 + y_0\cdot y_1 + \left(y_0\right)^2\right] \qquad I_y = \frac{t\cdot(s_1 - s_0)}{3}\cdot\left[\left(x_1\right)^2 + x_0\cdot x_1 + \left(x_0\right)^2\right]$$

$$I_x := \sum_i \frac{a_i}{3}\cdot\left[\left(y_{i+1}\right)^2 + y_i\cdot y_{i+1} + \left(y_i\right)^2\right] \qquad I_x = 833.333$$

$$I_y := \sum_i \frac{a_i}{3}\cdot\left[\left(x_{i+1}\right)^2 + x_i\cdot x_{i+1} + \left(x_i\right)^2\right] \qquad I_y = 31.25$$

$$I_{xy} = \frac{t\cdot(s_1 - s_0)}{6}\cdot\left[2\cdot\left(x_1\cdot y_1 + x_0\cdot y_0\right) + x_0\cdot y_1 + x_1\cdot y_0\right]$$

$$I_{xy} := \sum_i \frac{a_i}{6}\cdot\left[2\left(x_{i+1}\cdot y_{i+1} + x_i\cdot y_i\right) + x_i\cdot y_{i+1} + x_{i+1}\cdot y_i\right] \qquad I_{xy} = 0$$

$$I_{yx} := I_{xy}$$

calculate $\Delta\omega$c this is a running total: $\Delta\omega_c = \frac{x_1 + x_0}{2}\cdot\left(y_1 - y_0\right) - \frac{y_1 + y_0}{2}\cdot\left(x_1 - x_0\right)$

first calculate each increment $\Delta\omega c_i := \frac{x_{i+1} + x_i}{2}\cdot\Delta y_i - \frac{y_{i+1} + y_i}{2}\cdot\left(\Delta x_i\right)$

$$\omega c_0 := 0 \qquad \omega c_{i+1} := \omega c_i + \frac{x_{i+1} + x_i}{2}\cdot\Delta y_i - \frac{y_{i+1} + y_i}{2}\cdot\left(\Delta x_i\right) \qquad \omega c_{i+1} := \omega c_i + \Delta\omega c_i$$

calculate warping moments wrt centroid:

$$I_{y\omega c} = \frac{t\cdot(s_1 - s_0)}{6}\cdot\left[2\cdot\left(x_1\cdot\omega c_1 + x_0\cdot\omega c_0\right) + x_0\cdot\omega c_1 + x_1\cdot\omega c_0\right]$$

contribution from each segment

$$I_{x\omega c} = \frac{t\cdot(s_1 - s_0)}{6}\cdot\left[2\cdot\left(y_1\cdot\omega_{c1} + y_0\cdot\omega c_0\right) + y_0\cdot\omega c_1 + y_1\cdot\omega c_0\right]$$

$$I_{y\omega c} := \sum_i \frac{a_i}{6}\cdot\left[2\left(x_{i+1}\cdot\omega c_{i+1} + x_i\cdot\omega c_i\right) + x_i\cdot\omega c_{i+1} + x_{i+1}\cdot\omega c_i\right] \qquad I_{y\omega c} = 3.411\times 10^{-13}$$

$$I_{x\omega c} := \sum_i \frac{a_i}{6}\cdot\left[2\left(y_{i+1}\cdot\omega c_{i+1} + y_i\cdot\omega c_i\right) + y_i\cdot\omega c_{i+1} + y_{i+1}\cdot\omega c_i\right] \qquad I_{x\omega c} = 1.944\times 10^{3}$$

from torsion properties:

$$x_D := \frac{\left(I_{x\omega c} \cdot I_y - I_{xy} \cdot I_{y\omega c}\right)}{\left(I_x \cdot I_y - I_{xy}{}^2\right)} \qquad y_D := \frac{\left(-I_{y\omega c} \cdot I_x + I_{xy} \cdot I_{x\omega c}\right)}{\left(I_x \cdot I_y - I_{xy}{}^2\right)}$$

$$x_D = 2.333 \qquad y_D = -1.091 \times 10^{-14}$$

now we can calculate warping Ω relative to an arbitrary origin $\Omega 0 = 0$

$$\Delta\Omega_D(s) = \Delta\omega c - x_D \cdot \left(y_1 - y_0\right) + y_D \cdot \left(x_1 - x_0\right)$$

if we set $\Omega_{D0} = 0$ at the start of a line segment, then $\Omega_{D1} = \Omega_{D0} + \Delta\Omega_D$

$$\Delta\Omega_{D_i} := \Delta\omega c_i - x_D \cdot \Delta y_i + y_D \cdot \Delta x_i$$

$$\Omega_{D_0} := 0 \qquad \Omega_{D_{i+1}} := \Omega_{D_i} + \Delta\Omega_{D_i}$$

calculate "centroid" of warping wrt shear center:

$$\Delta Q_{\Omega D_i} := \frac{a_i}{2} \cdot \left(\Omega_{D_i} + \Omega_{D_{i+1}}\right) \qquad \Omega_{Dcg} := \frac{\sum \Delta Q_{\Omega D}}{A} \qquad \Omega_{Dcg} = -35 \qquad \omega_{D_j} := \Omega_{D_j} - \Omega_{Dcg}$$

now we can calculate the normalized warping functions (relative to the shear center)

$$I_\omega = \frac{t \cdot \left(s_1 - s_0\right)}{3} \cdot \left[\left(\omega D_1\right)^2 + \omega D_0 \cdot \omega D_1 + \left(\omega D_0\right)^2\right]$$

$$I_{y\omega D} = \frac{t \cdot \left(s_1 - s_0\right)}{6} \cdot \left[2 \cdot \left(x_1 \cdot \omega D_1 + x_0 \cdot \omega D_0\right) + x_0 \cdot \omega D_1 + x_1 \cdot \omega D_0\right]$$

$$I_{x\omega D} = \frac{t \cdot \left(s_1 - s_0\right)}{6} \cdot \left[2 \cdot \left(y_1 \cdot \omega_{D1} + y_0 \cdot \omega D_0\right) + y_0 \cdot \omega D_1 + y_1 \cdot \omega D_0\right]$$

$$I_{\omega} := \sum_i \frac{a_i}{3} \cdot \left[\left(\omega_{D_{i+1}}\right)^2 + \omega_{D_i} \cdot \omega_{D_{i+1}} + \left(\omega_{D_i}\right)^2 \right] \qquad I_{\omega} = 2.292 \times 10^{3}$$

$$I_{y\omega D} := \sum_i \frac{a_i}{6} \cdot \left[2\left(x_{i+1} \cdot \omega_{D_{i+1}} + x_i \cdot \omega_{D_i}\right) + x_i \cdot \omega_{D_{i+1}} + x_{i+1} \cdot \omega_{D_i} \right] \qquad I_{y\omega D} = -3.553 \times 10^{-13}$$

$$I_{x\omega D} := \sum_i \frac{a_i}{6} \cdot \left[2\left(y_{i+1} \cdot \omega_{D_{i+1}} + y_i \cdot \omega_{D_i}\right) + y_i \cdot \omega_{D_{i+1}} + y_{i+1} \cdot \omega_{D_i} \right] \qquad I_{x\omega D} = 9.379 \times 10^{-13}$$

Output:

$X_{cg} = 4.167$ $\qquad Y_{cg} = 10$

$x_D = 2.333$ $\qquad y_D = -1.091 \times 10^{-14}$

$\Omega_{Dcg} = -35$

$I_x = 833.333$ $\qquad I_{yx} = 0$

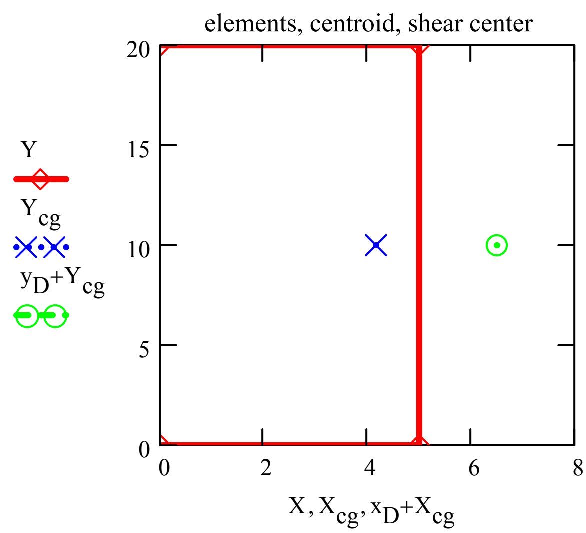
$I_y = 31.25$ $\qquad I_{xy} = 0$

$I_{x\omega c} = 1.944 \times 10^{3}$ $\qquad I_{y\omega c} = 3.411 \times 10^{-13}$

$I_{\omega} = 2.292 \times 10^{3}$

$I_{x\omega D} = 9.379 \times 10^{-13}$

$I_{y\omega D} = -3.553 \times 10^{-13}$



Note: the coordinate system in this plot is X, Y therefore xD and yD needs to have Xcg and Ycg added back in

first moment approach

to hold values from above $I_{xx} := I_x$

$I_{yy} := I_y$

repeat centroid calculations:

we will use these later $\Delta X_i := X_{i+1} - X_i$ $\Delta x_i := \Delta X_i$ $\Delta Y_i := Y_{i+1} - Y_i$ $\Delta y_i := \Delta Y_i$

calculate area if necessary

$$a_i := \text{if}\left[a_i = 0, t_i \cdot \sqrt{(\Delta X_i)^2 + (\Delta Y_i)^2}, a_i\right] \qquad A := \sum a \qquad A = 15$$

calculate centroid in X and Y coordinate system and coordinates in centroidal system:

$$\Delta Q_{X_i} := \frac{a_i}{2} \cdot (X_i + X_{i+1}) \qquad X_{cg} := \frac{\sum \Delta Q_X}{A} \qquad X_{cg} = 4.167 \qquad x_j := X_j - X_{cg}$$

$$\Delta Q_{Y_i} := \frac{a_i}{2} \cdot (Y_i + Y_{i+1}) \qquad Y_{cg} := \frac{\sum \Delta Q_Y}{A} \qquad Y_{cg} = 10 \qquad y_j := Y_j - Y_{cg}$$

first moment approach get midpoints and values for Q at end and midpoints:

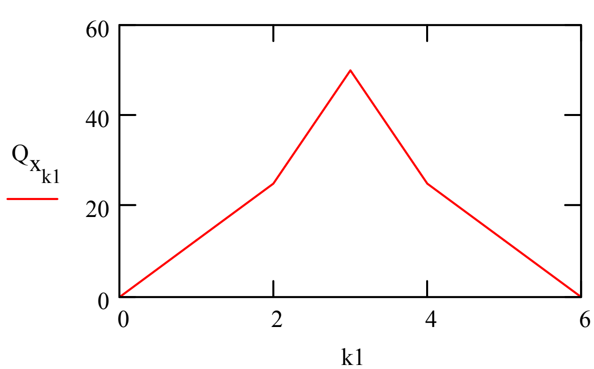
$$ym_i := \frac{y_{i+1} + y_i}{2} \qquad xm_i := \frac{x_{i+1} + x_i}{2}$$

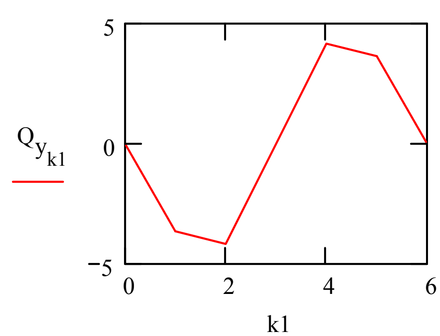
$$\Delta Q_{x_{2 \cdot i}} := \frac{a_i}{4} \cdot (y_i + ym_i) \quad \Delta Q_{x_{2 \cdot i+1}} := \frac{a_i}{4} \cdot (ym_i + y_{i+1}) \quad \Delta Q_{y_{2 \cdot i}} := \frac{a_i}{4} \cdot (x_i + xm_i) \quad \Delta Q_{y_{2 \cdot i+1}} := \frac{a_i}{4} \cdot (xm_i + x_{i+1})$$

$k := 1 .. 2 \cdot n\_elements$ we have a total of 2*n_elements + 1, k = 1...2*n_elements and 0

$$Q_{x_0} := 0 \qquad Q_{x_k} := \left(Q_{x_{k-1}} + \Delta Q_{x_{k-1}}\right) \qquad Q_{y_0} := 0 \qquad Q_{y_k} := \left(Q_{y_{k-1}} + \Delta Q_{y_{k-1}}\right)$$

$k1 := 0 .. 2 \cdot n\_elements + 1$





integrate using Simpson's rule with midpoint values

$$Q_{x\_bar_i} := Q_{x_{2\cdot i}} + 4\cdot Q_{x_{2\cdot i+1}} + Q_{x_{2\cdot(i+1)}}$$

$$Q_{y\_bar_i} := Q_{y_{2\cdot i}} + 4\cdot Q_{y_{2\cdot i+1}} + Q_{y_{2\cdot(i+1)}}$$

$$I_x := \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q_{x\_bar_i}\cdot\Delta y_i$$

$$I_y := \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q_{y\_bar_i}\cdot\Delta x_i$$

$I_x = 833.333$ $\quad I_{xx} = 833.333$ $\quad I_y = 31.25$ $\quad I_{yy} = 31.25$

cross moments of inertia

$$I_{xy} := \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q_{x\_bar_i}\cdot\Delta x_i$$

$$I_{yx} := \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q_{y\_bar_i}\cdot\Delta y_i$$

$I_{xy} = 0$ $\quad I_{yx} = -5.329\times 10^{-14}$

warping moments relative to the centroid:

$$\Delta\omega_{c_i} := xm_i\cdot\Delta y_i - ym_i\cdot\Delta x_i \qquad I_{x\omega c} := \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q_{x\_bar_i}\cdot\Delta\omega_{c_i} \qquad I_{y\omega c} := \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q_{y\_bar_i}\cdot\Delta\omega_{c_i}$$

$I_{x\omega c} = 1.944\times 10^{3}$ $\quad I_{y\omega c} = -3.032\times 10^{-13}$

as above calculate shear center

$$x_D := \frac{\left(I_{x\omega c}\cdot I_y - I_{xy}\cdot I_{y\omega c}\right)}{\left(I_x\cdot I_y - I_{xy}^{2}\right)} \qquad x_D = 2.333 \qquad y_D := \frac{\left(-I_{y\omega c}\cdot I_x + I_{xy}\cdot I_{x\omega c}\right)}{\left(I_x\cdot I_y - I_{xy}^{2}\right)} \qquad y_D = 9.701\times 10^{-15}$$

now as above we can calculate the warping parmeters

now we can calculate warping Ω relative to an arbitrary origin $\Omega 0 = 0$

$$\Delta\Omega_D(s) = \Delta\omega_c - x_D\cdot\left(y_1 - y_0\right) + y_D\cdot\left(x_1 - x_0\right)$$

if we set $\Omega_{D0} = 0$ at the start of a line segment, then $\Omega_{D1} = \Omega_{D0} + \Delta\Omega_D$

$$\Delta\Omega_{D_i} := \Delta\omega_{c_i} - x_D\cdot\Delta y_i + y_D\cdot\Delta x_i \qquad \Omega_{D_0} := 0 \qquad \Omega_{D_{i+1}} := \Omega_{D_i} + \Delta\Omega_{D_i}$$

calculate "centroid" of warping wrt shear center:

$$\Delta Q_{\Omega D_i} := \frac{a_i}{2}\cdot\left(\Omega_{D_i} + \Omega_{D_{i+1}}\right) \qquad \Omega_{Dcg} := \frac{\sum \Delta Q_{\Omega D}}{A} \qquad \Omega_{Dcg} = -35 \qquad \omega_j := \Omega_{D_j} - \Omega_{Dcg}$$

$$\omega m_i := \frac{\omega_i + \omega_{i+1}}{2}$$

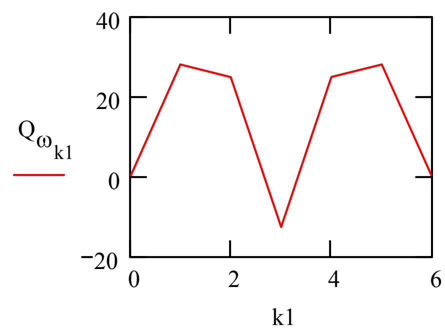
$$\Delta Q_{\omega_{2\cdot i}} := \frac{a_i}{4}\cdot\left(\omega_i + \omega m_i\right) \qquad \Delta Q_{\omega_{2\cdot i+1}} := \frac{a_i}{4}\cdot\left(\omega m_i + \omega_{i+1}\right)$$

$$k := 1 .. 2\cdot n\_elements \qquad Q_{\omega_0} := 0$$

$$Q_{\omega_k} := Q_{\omega_{k-1}} + \Delta Q_{\omega_{k-1}} \qquad Q_{\omega\_bar_i} := Q_{\omega_{2\cdot i}} + 4\cdot Q_{\omega_{2\cdot i+1}} + Q_{\omega_{2\cdot(i+1)}}$$

$$\Delta\omega_i := \omega_{i+1} - \omega_i$$

$$I_{\omega\omega} := \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q_{\omega\_bar_i}\cdot\Delta\omega_i \qquad I_{\omega\omega} = 2.292\times 10^{3}$$



the cross moments are:

$$I_{x\omega} := \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q_{x\_bar_i}\cdot\Delta\omega_i \qquad I_{x\omega} = 7.579\times 10^{-14}$$

$$I_{y\omega} := \frac{-1}{6}\cdot\sum_{i=0}^{n\_elements-1} Q_{y\_bar_i}\cdot\Delta\omega_i \qquad I_{y\omega} = 1.137\times 10^{-13}$$

Output:

$X_{cg} = 4.167$ $Y_{cg} = 10$

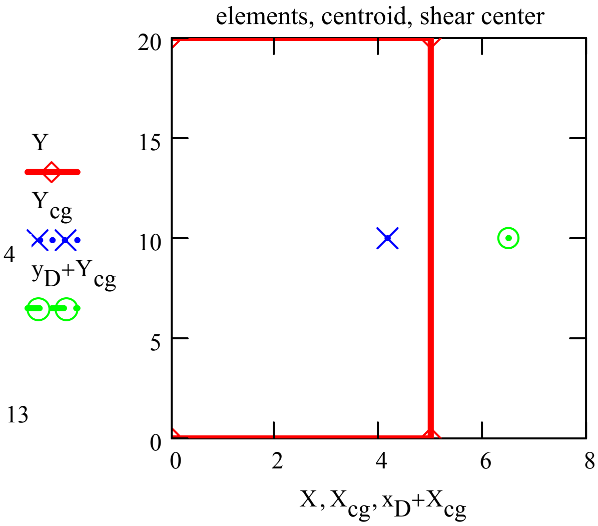
$x_D = 2.333$ $y_D = 9.701 \times 10^{-15}$

$\Omega_{Dcg} = -35$

$I_x = 833.333$ $I_{yx} = -5.329 \times 10^{-14}$

$I_y = 31.25$ $I_{xy} = 0$

$I_{x\omega c} = 1.944 \times 10^{3}$ $I_{y\omega c} = -3.032 \times 10^{-13}$



$I_\omega = 2.292 \times 10^{3}$

$I_{x\omega} = 7.579 \times 10^{-14}$

$I_{y\omega} = 1.137 \times 10^{-13}$

Note: the coordinate system in this plot is X, Y therefore xD and yD needs to have Xcg and Ycg added back in

if the example is as in the starting point:

the results can be compared with Shames: example 11.20

$$X := \begin{pmatrix} 0 \\ 5 \\ 5 \\ 0 \end{pmatrix} \qquad Y := \begin{pmatrix} 20 \\ 20 \\ 0 \\ 0 \end{pmatrix} \qquad t := \begin{pmatrix} 0.5 \\ 0.5 \\ 0.5 \end{pmatrix}$$

$$e = \frac{t_1 \cdot b^2}{2 \cdot b \cdot t_1 + t_2 \cdot \frac{h}{3}}$$

for channel shape where e = distance from web as shown

$t_{flange} := t_0$ $t_{web} := t_1$

$t_1 := t_{flange}$ $t_2 := t_{web}$

$b := 5$
$h := 20$

$$e := \frac{t_1 \cdot b^2}{2 \cdot b \cdot t_1 + t_2 \cdot \frac{h}{3}}$$

$e = 1.5$ from web as defined

compares to distance from cg

$e_{rel\_cg} := e + (5 - X_{cg})$

$X_{cg} = 4.167$

$x_D = 2.333$

$e_{rel\_cg} = 2.333$