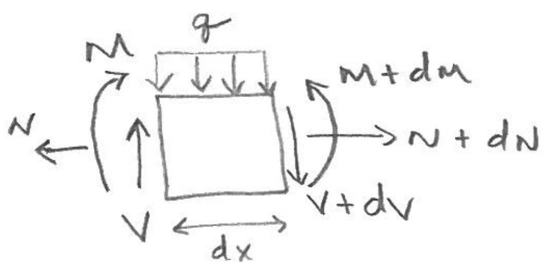
Recitation 3

3.1 Summary of Beam Equations

Equilibrium:



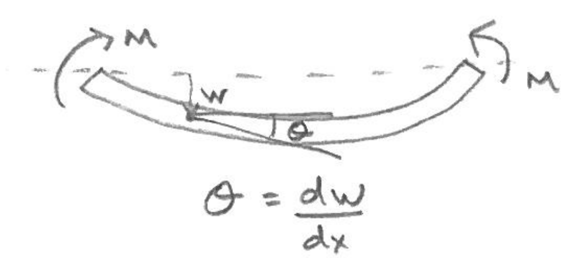
$$\frac{\mathrm{d}N}{\mathrm{d}x} = 0 \tag{3.1}$$

$$\left.\begin{aligned} \frac{\mathrm{d}V}{\mathrm{d}x} + g = 0 \\ \frac{\mathrm{d}M}{\mathrm{d}x} = V \end{aligned}\right\} \frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + g = 0 \tag{3.2}$$

Hooke's Law:

$$M = EI\kappa \tag{3.3}$$

Geometry:



$$\kappa = \frac{\mathrm{d}\theta}{\mathrm{d}x} = -\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{3.4}$$

$$M = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2} \tag{3.5}$$

$$V = -EI\frac{\mathrm{d}^3 w}{\mathrm{d}x^3} \tag{3.6}$$

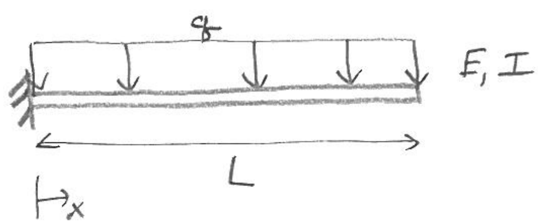
$$\frac{\mathrm{d}^2 M}{\mathrm{d}x^2} + q = 0 \rightarrow \boxed{EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q} \tag{3.7}$$

3.2 Methods of Solution

1. Direct Integration: $EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q$
2. Uncoupled solution: find $M(x)$, then integrate $M(x) = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$
3. Rayleigh-Ritz:
   - Assume shape function
   - Apply BC's
   - Calclate $\Pi = U - V$, find $w(x)$ to minimize Π
4. Castigliano's Theorem: $w = \frac{\partial U}{\partial P}$

Example

Find the max. deflection for the following cantilever beam:



Direct integration: $EI\frac{\mathrm{d}^4 w}{\mathrm{d}x^4} = q$

$$\text{1st integ: } \frac{\mathrm{d}^3 w}{\mathrm{d}x^3} = \frac{1}{EI}(qx + C_1) \tag{3.8}$$

$$\text{2nd integ: } \frac{\mathrm{d}^2 w}{\mathrm{d}x^2} = \frac{1}{EI}(q\frac{x^2}{2} + C_1 x + C_2) \tag{3.9}$$

$$\text{3rd integ: } \frac{\mathrm{d}w}{\mathrm{d}x} = \frac{1}{EI}(q\frac{x^3}{6} + C_1\frac{x^2}{2} + C_2 x + C_3) \tag{3.10}$$

$$\text{4th integ: } w = \frac{1}{EI}(q\frac{x^4}{24} + C_1\frac{x^3}{6} + C_2\frac{x^2}{2} + C_3 x + C_4) \tag{3.11}$$

Need 4 BC's to evaluate C_1, C_2, C_3, and C_4

$$①\, w(0) = 0 \to C_4 = 0 \tag{3.12}$$

$$②\, w'(0) = 0 \to C_3 = 0 \tag{3.13}$$

$$③\, M(L) = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}(L) = 0 \quad \to \quad q\frac{L^2}{2} + C_1 L + C_2 = 0 \to C_2 = -q\frac{L^2}{2} - C_1 L \tag{3.14}$$

$$④\, V(L) = -EI\frac{d^3 w}{dx^3}(L) = 0 \to qL + C_1 = 0 \to C_1 = -qL \tag{3.15}$$

$$C_2 = -\frac{qL^2}{2} + qL^2 = \frac{qL^2}{2} \tag{3.16}$$

So

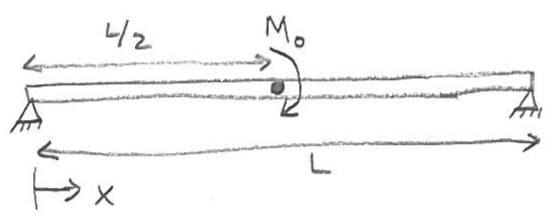
$$w(x) = \frac{1}{EI}\left(q\frac{x^4}{24} - \frac{qLx^3}{6} + \frac{qL^2x^2}{4}\right) \tag{3.17}$$

Max. at

$$x = L \to w(L) = \frac{q}{EI}\left(\frac{L^4}{24} - \frac{L^4}{6} + \frac{L^4}{4}\right) = \frac{3qL^4}{24EI} = \boxed{\frac{qL^4}{8EI}} \tag{3.18}$$

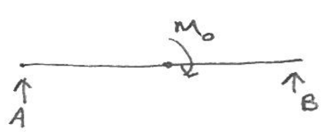
Example

Find $w(x)$ for the following beam:



Find $M(x)$, then use $M(x) = -EI\frac{\mathrm{d}^2 w}{\mathrm{d}x^2}$:

First find reaction forces:

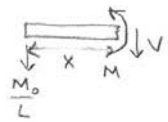


$$+\Sigma M_A = 0{:}\ -M_0 + BL = 0 \to B = \frac{M_0}{L} \tag{3.19}$$

$$\Sigma F_y = 0{:}\ A + B = 0 \to A = -B = \frac{M_0}{L} \tag{3.20}$$

— $M(x)$ will be discontinuous at $L/2 \to$ need to evaluate both sections.

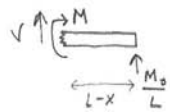
For $x < L/2$:



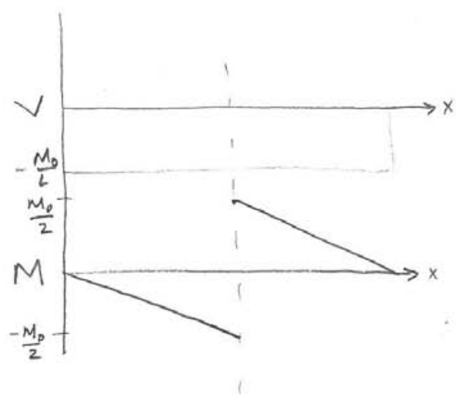
$$V = -\frac{M_0}{L}$$
$$M = -\frac{M_0}{L}x$$

SIGN CONVENTION

For $x > L/2$:



$$V = -\frac{M_0}{L}$$
$$M = \frac{M_0}{L}(L - x) = M_0 - \frac{M_0}{L}x$$



$$M(x) = \begin{cases} -\dfrac{M_0}{l}x & (x < L/2) \\ M_0 - \dfrac{M_0}{l}x & (x > L/2) \end{cases} \tag{3.21}$$

— $M(x) = -EI\dfrac{\mathrm{d}^2w}{\mathrm{d}x^2} \rightarrow$ Ingegrate twice to get $w(x)$:

For $x < L/2$: $-\dfrac{M_0}{l}x = -EI\dfrac{\mathrm{d}^2w}{\mathrm{d}x^2}$

1st int: $\dfrac{\mathrm{d}w}{\mathrm{d}x} = \dfrac{M_0}{EIL}\dfrac{x^2}{2} + C_1$

2nd int: $w = \dfrac{M_0}{EIL}\dfrac{x^3}{6} + C_1x + C_2$

For $x > L/2$: $M_0 - \dfrac{M_0}{l}x = -EI\dfrac{\mathrm{d}^2w}{\mathrm{d}x^2}$

1st int: $\dfrac{\mathrm{d}w}{\mathrm{d}x} = -\dfrac{M_0x}{EI} + \dfrac{M_0}{EIL}\dfrac{x^2}{2} + C_3$

2nd int: $w = -\dfrac{M_0}{EI}\dfrac{x^2}{2} + \dfrac{M_0}{EIL}\dfrac{x^3}{6} + C_3x + C_4$

— We need *4 BC's* to evaluate C_1, C_2, C_3, and C_4.

① $w(0) = 0 \rightarrow C_2 = 0$

② $w(L) = 0 \rightarrow -\dfrac{M_0}{EI}\dfrac{L^2}{2} + \dfrac{M_0}{EIL}\dfrac{L^3}{6} + C_3L + C_4 = 0 \rightarrow C_4 = \dfrac{M_0L^2}{3EI} - C_3L$

③ $\dfrac{dw}{dx}$ at $\dfrac{L}{2}$ must be *continuous*

$$\frac{M_0}{EIL}\frac{(L/2)^2}{2} + C_1 = -\frac{M_0(L/2)}{EI} + \frac{M_0}{EIL}\frac{(L/2)^2}{2} + C_3 \rightarrow C_1 = C_3 - \frac{M_0 L}{2EI}$$

④ w at $L/2$ must be *continuous*

$$\frac{M_0}{EIL}\frac{(L/2)^3}{6} + C_1(\frac{L}{2}) = -\frac{M_0}{EI}\frac{(L/2)^2}{2} + \frac{M_0}{EIL}\frac{(L/2)^3}{6} + C_3(\frac{L}{2}) + \underbrace{\frac{M_0L^2}{3EI} - C_3 L}_{C_4}$$

— Substitute for C_1, solve for C_3:

$$(C_3 - \frac{M_0 L}{2EI})\frac{L}{2} = -C_3(\frac{L}{2}) + \frac{5}{24}\frac{M_0 L^2}{EI}$$

$$\boxed{C_3 = \frac{11}{24}\frac{M_0 L}{EI}}$$

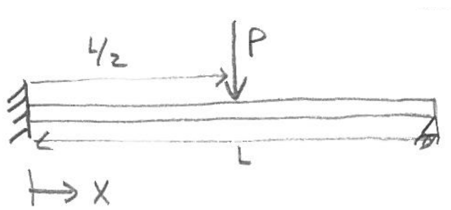
$$C_1 = \frac{11}{24}\frac{M_0 L}{EI} - \frac{M_0 L}{2EI} = \boxed{-\frac{M_0 L}{24EI}}$$

$$C_4 = \frac{M_0 L^2}{3EI} - \frac{11}{24}\frac{M_o L^2}{EI} = \boxed{-\frac{M_0 L^2}{8EI}}$$

$$\Longrightarrow \quad \boxed{w(x) = \begin{cases} \dfrac{M_0}{6EIL}x^3 - \dfrac{M_0 L}{24EI}x & (x \leqslant L/2) \\ \dfrac{M_0}{6EIL}x^3 - \dfrac{M_0}{2EI}x^2 + \dfrac{11}{24}\dfrac{M_0 L}{EI}x - \dfrac{M_0 L^2}{8EI} & (x \geqslant L/2) \end{cases}} \qquad (3.22)$$

Example

Find $w(x)$ for the following beam:



— Statically indeterminant → must use direct integration

	$x < L/2$	$x > L/2$
	$\dfrac{d^4w}{dx^4} = 0$	$\dfrac{d^4w}{dx^4} = 0$
1st int:	$\dfrac{d^3w}{dx^3} = C_1$	$\dfrac{d^3w}{dx^3 = C_5}$
2nd int:	$\dfrac{d^2w}{dx^2} = C_1x + C_2$	$\dfrac{d^2w}{dx^2} = C_5x + C_6$
3rd int:	$\dfrac{dw}{dx} = \dfrac{C_1}{2}x^2 + C_2x + C_3$	$\dfrac{dw}{dx} = \dfrac{C_5}{2}x^2 + C_6x + C_7$
4th int:	$w = \dfrac{C_1}{6}x^3 + \dfrac{C_2}{2}x^2 + C_3x + C_4$	$w = \dfrac{C_5}{6}x^3 + \dfrac{C_6}{2}x^2 + C_7x + C_8$

— We need *8 BS's* to evaluate C_1–C_8:

① $w(0) = 0 \rightarrow \boxed{C_4 = 0}$

② $w'(0) = 0 \rightarrow \boxed{C_3 = 0}$

③ $M(L) = -EI\dfrac{d^2w}{dx^2}(L) = 0 \rightarrow C_5L + C_6 = 0$ ⓐ

④ $w(L) = 0 \rightarrow \dfrac{C_5}{6}L^3 + \dfrac{C_6}{2}L^2 + C_7L + C_8 = 0$ ⓑ

⑤ $w(L/2)$ is continuous: $\dfrac{C_1}{6}(\dfrac{L}{2})^3 + \dfrac{C_2}{2}(\dfrac{L}{2})^2 = \dfrac{C_5}{6}(\dfrac{L}{2})^3 + \dfrac{C_6}{2}(\dfrac{L}{2})^2 + C_7(\dfrac{L}{2}) + C_8$ ⓒ

⑥ $w'(L/2)$ is continuous: $\dfrac{C_1}{2}(\dfrac{L}{2})^2 + C_2(\dfrac{L}{2}) = \dfrac{C_5}{2}(\dfrac{L}{2})^2 + C_6(\dfrac{L}{2}) + C_7$ ⓓ

⑦ $M(L/2)$ is continuous: $C_1(\dfrac{L}{2}) + C_2 = C_5(\dfrac{L}{2}) + C_6$ ⓔ

⑧ $V(\dfrac{L}{2}^-) = V(\dfrac{L}{2}^+) + P \rightarrow -EIC_1 - EIC_5 + P$ ⓕ

(Step jump due to point load)

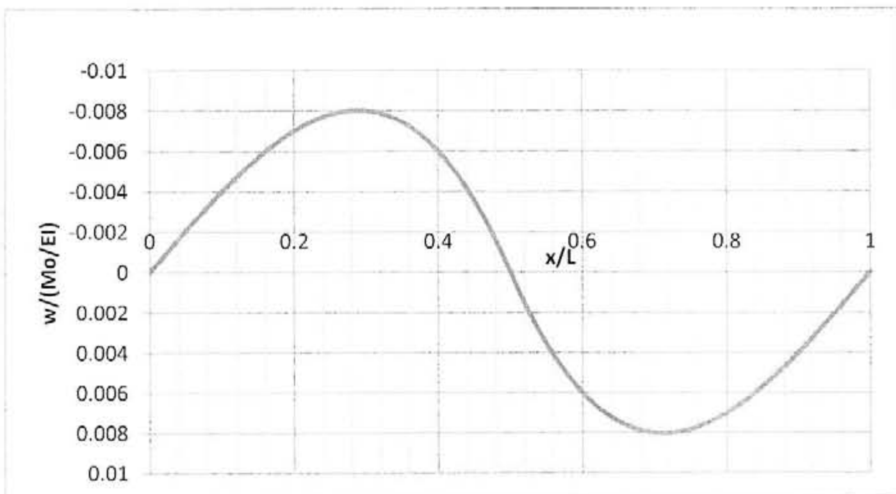
— Now we have 6 eons (ⓐ-ⓕ) to solve for C_1, C_2, C_5-C_8

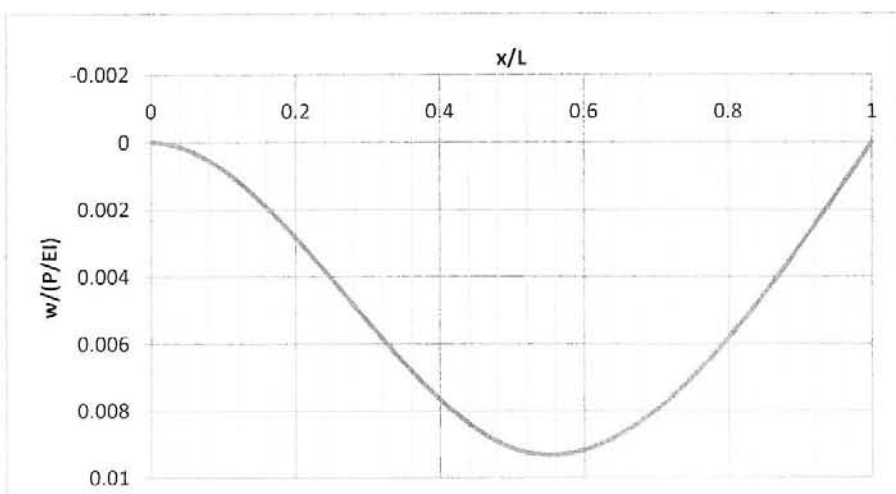
(Omit algebra)

$$C_1 = -\frac{11}{16}\frac{P}{EI},\ C_2 = \frac{3}{16}\frac{PL}{EI},\ C_5 = \frac{5}{16}\frac{P}{EI},\ C_6 = -\frac{5}{16}\frac{PL}{EI},\ C_7 = \frac{PL^2}{8EI},\ C_8 = -\frac{PL^3}{48EI}$$

So

$$\boxed{w(x) = \begin{cases} \dfrac{P}{EI}\left(-\dfrac{11}{96}x^3 + \dfrac{3}{32}x^2 L\right) & (x \leqslant L/2) \\ \dfrac{P}{EI}\left(\dfrac{5}{96}x^3 - \dfrac{5}{32}x^2 L + \dfrac{1}{8}xL^2 - \dfrac{1}{48}L^3\right) & (x \leqslant L/2) \end{cases}} \tag{3.23}$$





2.080J / 1.573J Structural Mechanics
Fall 2013