

**1.138J/2.062J, WAVE PROPAGATION**

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## Chapter 1. SAMPLE WAVE PROBLEMS

To describe a problem in mathematical terms, one must make use of the basic laws that govern the elements of the problem. In continuum mechanics, these are the conservation laws for mass and momentum. In addition, empirical constitutive laws are often needed to relate certain unknown variables; examples are equations of state, Hooke's law between stress and strain, etc.

To derive the conservation law one may consider an infinitesimal element (a line segment, area or volume element), yielding a differential equation directly. Alternately, one may consider a control volume (or area, or line segment) of arbitrary size in the medium of interest. The law is first obtained in integral form; a differential equation is then derived by using the arbitrariness of the control volume. The two approaches are completely equivalent.

Let us first demonstrate the differential approach.

**2 Transverse vibration of a taut string**

Referring to Figure 1, consider a taut string stretched between two fixed points at  $x = 0$  and  $x = L$ . Let the cross-sectional area be  $S$ . If there is an initial stretching of  $\Delta L$ , the initial tension  $T$  must be

$$T = ES \frac{\Delta L}{L}$$

by Hooke's law, where  $E$  is Young's modulus.

Now study the lateral displacement of the string from the initial position. By the law of conservation of transverse momentum, the total lateral force on the string element must be balanced by its inertia. Let the lateral displacement be  $V(x, t)$  and consider a differential element between  $x$  and  $x + dx$ . The net transverse force due to the difference of tension at both ends of the element is

$$(T \sin \alpha)_{x+dx} - (T \sin \alpha)_x,$$

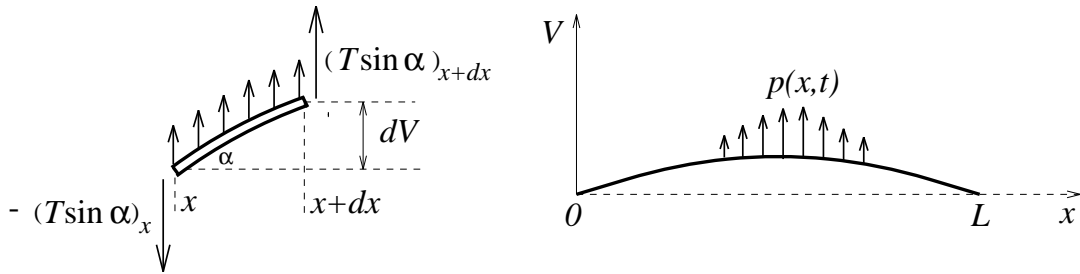


Figure 1: Deformation of a taut string

where

$$\sin \alpha = \frac{dV}{\sqrt{dx^2 + dV^2}} = \frac{\frac{\partial V}{\partial x}}{\sqrt{1 + \left(\frac{\partial V}{\partial x}\right)^2}}.$$

We shall assume the displacement to be small everywhere so that the slope is also small:  $\frac{\partial V}{\partial x} \ll 1$ . The local value of  $\sin \alpha$  can then be approximated by

$$\frac{\partial V}{\partial x} + O\left(\frac{\partial V}{\partial x}\right)^3,$$

where the expression  $O(\delta)$  stands for *of the order of*  $\delta$ . For any smooth function  $f$ , Taylor expansion gives

$$f(x + dx) - f(x) = \left(\frac{\partial f}{\partial x}\right) dx + O(dx)^2,$$

where the derivative is evaluated at  $x$ . Hence the net tension is

$$\frac{\partial}{\partial x} \left( T \frac{\partial V}{\partial x} \right) dx + O(dx)^2.$$

The instantaneous length  $\ell(x, t)$  of the string from 0 to  $x$  is

$$\ell(x, t) = \int_0^x dx \left[ 1 + \left(\frac{\partial V}{\partial x}\right)^2 \right]^{1/2} = x \left[ 1 + O\left(\frac{\partial V}{\partial x}\right)^2 \right].$$

It follows that

$$\frac{\ell - x}{x} = O\left(\frac{\partial V}{\partial x}\right)^2 \quad \text{for all } 0 < x < L,$$

which is of second-order smallness. The string length, hence the tension, is essentially unchanged with an error of  $O(\partial V/\partial x)^2$ , i.e.,  $T$  can be taken as constant with a similarly

small error. Thus the net tension in the string element is well represented by

$$T \frac{\partial^2 V}{\partial x^2} dx.$$

If the mass per unit length of the string is  $\rho$ , the inertia of the element is  $\rho(\partial^2 V/\partial t^2)dx$ . Let the applied load per unit length be  $p(x, t)$ . Momentum conservation requires that

$$\rho dx \frac{\partial^2 V}{\partial t^2} = T \frac{\partial^2 V}{\partial x^2} dx + p dx + O(dx)^2.$$

Eliminating  $dx$  and taking the limit of  $dx \rightarrow 0$ , we get

$$\frac{\rho}{T} \frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = \frac{p}{T}. \quad (2.1)$$

This equation, called the *wave equation*, is a partial differential equation of the second order. It is linear in the unknown  $V$  and inhomogeneous because of the forcing term on the right-hand side.

Is the longitudinal displacement  $U$  important in this problem? Conservation of momentum in the  $x$  direction requires that

$$\rho dx \frac{\partial^2 U}{\partial t^2} = (T \cos \alpha)_{x+dx} - (T \cos \alpha)_x.$$

Since

$$\cos \alpha = \frac{dx}{\sqrt{(dx)^2 + (dV)^2}} = \frac{1}{\sqrt{1 + \left(\frac{\partial V}{\partial x}\right)^2}} \cong 1 + O\left(\frac{\partial V}{\partial x}\right)^2,$$

the acceleration is of second-order smallness

$$\frac{\rho}{T} \frac{\partial^2 U}{\partial t^2} = O\left(\frac{\partial}{\partial x} \left(\frac{\partial V}{\partial x}\right)^2\right) = O\left(\left(\frac{\partial V}{\partial x}\right) \frac{\rho}{T} \frac{\partial^2 V}{\partial t^2}\right).$$

Hence  $U = O\left(\frac{\partial V}{\partial x}\right)V$  by twice integration with respect to  $t$ , and the longitudinal displacement can be ignored.

The differential equation (2.1) involves second-order derivatives with respect to both  $x$  and  $t$ . Two auxiliary conditions are needed for each variable. For example, at the initial instant, we may prescribe both the displacement and the velocity:

$$V(x, 0) = f(x) \quad (2.2)$$

and

$$\frac{\partial V}{\partial t}(x, 0) = g(x). \quad (2.3)$$

These statements are called the *initial conditions*. In addition we must also specify the *boundary conditions* at the ends. For a string stretched between two fixed ends, we require

$$V(0, t) = 0 \quad \text{and} \quad V(L, t) = 0. \quad (2.4)$$

Together with the partial differential equation, these auxiliary conditions define the initial-boundary-value problem. From the mathematical point of view, it is important to establish whether such a problem is well posed. This question involves the proof for the existence, uniqueness and stability of the solution.

As seen in this example, Taylor expansion is used at almost every step of the derivation. Indeed, it is indispensable not only in deriving governing equations, but also in obtaining approximate solutions of the equations, and in analyzing the physical content of the solution.

Note that the dimension of the coefficient  $T/\rho$  is

$$\left[ \frac{T}{\rho} \right] = \frac{ML/t^2}{M/L} = \left( \frac{L}{t} \right)^2 = [\text{velocity}]^2.$$

Now introduce the notation  $c = \sqrt{T/\rho}$ , which is a characteristic velocity of the physical problem. Equation (2.1) can then be written

$$\frac{1}{c^2} \frac{\partial^2 V}{\partial t^2} - \frac{\partial^2 V}{\partial x^2} = \frac{p}{T}, \quad (2.5)$$

which is called the *wave equation* arising in numerous contexts.

### 3 Longitudinal vibration of an elastic rod

Consider an elastic rod with the cross-sectional area  $S(x)$  and Young's modulus  $E$ , as shown in Figure (2). Let the longitudinal displacement from equilibrium be  $U(x, t)$ .

The strain at station  $x$  is

$$\lim_{\Delta x \rightarrow 0} \frac{\Delta U}{\Delta x} = \frac{\partial U}{\partial x}.$$

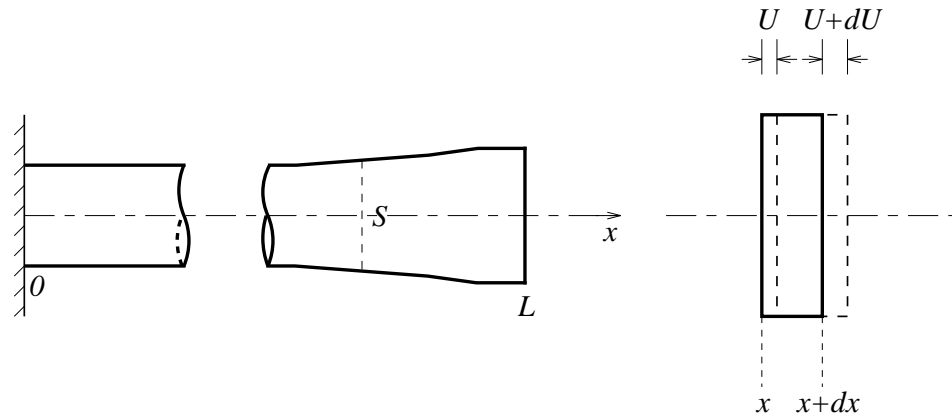


Figure 2: Longitudinal deformation of an elastic rod

By Hooke's law, the tension at  $x$  is

$$ES \frac{\partial U}{\partial x}.$$

Now the net tension on a rod element from  $x$  to  $x + dx$  is

$$\left( ES \frac{\partial U}{\partial x} \right)_{x+dx} - \left( ES \frac{\partial U}{\partial x} \right)_x = dx \frac{\partial}{\partial x} \left( ES \frac{\partial U}{\partial x} \right) + O(dx)^2.$$

Let the externally applied longitudinal force be  $f(x, t)$  per unit length. Momentum conservation requires that

$$\rho S \frac{\partial^2 U}{\partial t^2} dx = \frac{\partial}{\partial x} \left( ES \frac{\partial U}{\partial x} \right) dx + f dx + O(dx)^2.$$

In the limit of vanishing  $dx$ , we get the differential equation:

$$\rho S \frac{\partial^2 U}{\partial t^2} = \frac{\partial}{\partial x} \left( ES \frac{\partial U}{\partial x} \right) + f. \quad (3.1)$$

In the special case of uniform cross section,  $S = \text{constant}$ , and zero external force, then  $U$  satisfies the inhomogeneous wave equation

$$\frac{1}{c^2} \frac{\partial^2 U}{\partial t^2} = \frac{\partial^2 U}{\partial x^2} + \frac{f}{ES}, \quad (3.2)$$

where  $c = \sqrt{E/\rho}$  has the dimension of velocity.

The simplest boundary conditions are for fixed or free ends. If both ends are fixed, then,

$$U(0, t) = 0 \quad \text{and} \quad U(L, t) = 0. \quad (3.3)$$

If the left end is fixed but the right end is free, then

$$U(0, t) = 0 \quad \text{and} \quad \frac{\partial U}{\partial x}(L, t) = 0, \quad (3.4)$$

since the stress is proportional to the strain. Again, the most natural initial conditions are

$$U(x, 0) = f(x), \quad \text{and} \quad \frac{\partial U}{\partial t}(x, 0) = g(x), \quad (3.5)$$

where  $f$  and  $g$  are prescribed functions of  $x$  for  $0 < x < L$ .

Let us change to the integral approach in the next example.

## 4 Traffic flow on a freeway

One of the mathematical models of traffic flow is the hydrodynamical theory of Lighthill and Whitham (1958). It is a simple theory capable of describing many real-life features of highway traffic with remarkable faithfulness. Consider any section of a straight freeway from  $x = a$  to  $x = b$ , Figure 3. Assume for simplicity that there are no exits or entrances, and all vehicles are on the go. Let the density of cars (number of cars per unit length of highway) at  $x$  and  $t$  be  $\rho(x, t)$ , and the flux of cars (number of cars crossing the point  $x$  per unit time) be  $q(x, t)$ . By requiring that the number of cars within an arbitrary section from  $a$  to  $b$  be conserved, we have

$$-\frac{\partial}{\partial t} \int_a^b \rho(x, t) dx = q(b, t) - q(a, t) .$$

Rewriting the right-hand side

$$q(b, t) - q(a, t) = \int_a^b \frac{\partial q}{\partial x} dx ,$$

we get

$$\int_a^b \left( \frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} \right) dx = 0 . \quad (4.1)$$

Since the control interval  $(a, b)$  is arbitrary, the integrand must vanish,

$$\frac{\partial \rho}{\partial t} + \frac{\partial q}{\partial x} = 0. \quad (4.2)$$

This result can be argued by contradiction, which is a typical reasoning needed to change an integral law to a differential law. Suppose that the integrand is positive somewhere

within  $(a, b)$ , say, in the range  $(a', b') \in (a, b)$ , and zero elsewhere in  $(a, b)$ , then the integral in (1.3.1) must be positive. But this is a contradiction. The assumption that the integrand is positive somewhere is therefore wrong. By a similar argument, the integrand cannot be negative anywhere, and hence must be zero everywhere in  $(a, b)$ .

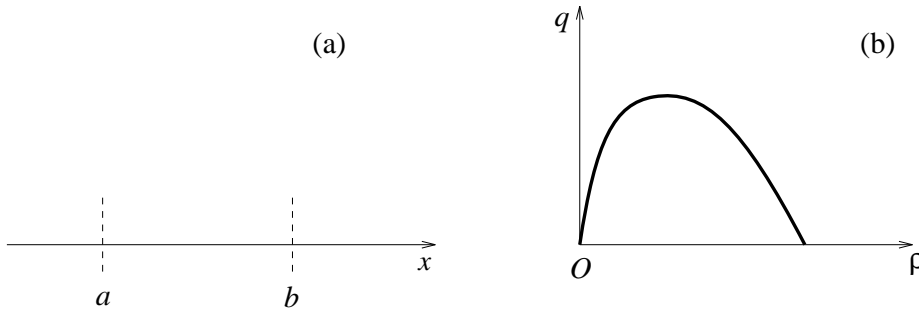


Figure 3: (a). A section of the freeway. (b). The relation between traffic flux rate and traffic density.

Equation (1.3.2) is the law of conservation of cars. Having two unknowns  $q$  and  $\rho$ , a constitutive relation between  $\rho$  and  $q$  is needed and must be found by field measurements. Heuristically,  $q$  must be zero when there is no car on the road, and zero again when the density attains a maximum (bumper-to-bumper traffic), hence the relation between  $q$  and  $\rho$  must be nonlinear

$$q = q(\rho) \quad (4.3)$$

as sketched in Figure 1.4.b. With this relation, (1.3.2) becomes

$$\frac{\partial \rho}{\partial t} + \left( \frac{dq}{d\rho} \right) \frac{\partial \rho}{\partial x} = 0. \quad (4.4)$$

This result is a first-order nonlinear partial differential equation and will be used to deduce a variety of interesting phenomena of traffic flow.

In all the examples studied so far the final governing equation involves only one unknown. Now we will examine a problem with several unknowns.

## 5 Wave propagation in arteries

We shall examine the pulsating flow of blood in an artery whose wall is thin and elastic. As a first exercise let us assume that there is only pulsation but no net flux. Because of the pressure gradient in the blood, the artery wall must deform. The elastic restoring force in the wall makes it possible for waves to propagate.

The artery radius  $a(x, t)$  varies from the constant mean  $a_o$  in time and along the artery (in  $x$ ). Let the local cross sectional area be  $S = \pi a^2$ , and the averaged velocity be  $u(x, t)$ . Consider a fixed geometrical volume between  $x$  and  $x + dx$ , through which fluid moves in and out. Conservation of mass requires

$$\frac{\partial S}{\partial t} + \frac{\partial(uS)}{\partial x} = 0, \quad (5.1)$$

Next the momentum balance. The time rate of momentum change in the volume must be balanced by the net influx of momentum through the two ends and the pressure force acting on all sides. The rate of momentum change is

$$\frac{\partial(\rho u S)}{\partial t} \quad (5.2)$$

The net rate of momentum influx is

$$-\frac{\partial(\rho u^2 S)}{\partial x} dx = -\rho u \frac{\partial u S}{\partial x} - \rho u S \frac{\partial u}{\partial x} \quad (5.3)$$

The net pressure force at the two ends is

$$-\frac{\partial(pS)}{\partial x} = -S \frac{\partial p}{\partial x} - p \frac{\partial S}{\partial x}$$

while that on the sloping wall is

$$2\pi a p \frac{\partial a}{\partial x} = p \frac{\partial S}{\partial x}$$

The sum of all pressure forces is

$$-S \frac{\partial p}{\partial x} \quad (5.4)$$

Balancing the momentum by equating (5.2) to the sum of (5.3) and (5.4) we get, after making use of mass conservation (5.1),

$$\rho \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{\partial p}{\partial x} \quad (5.5)$$



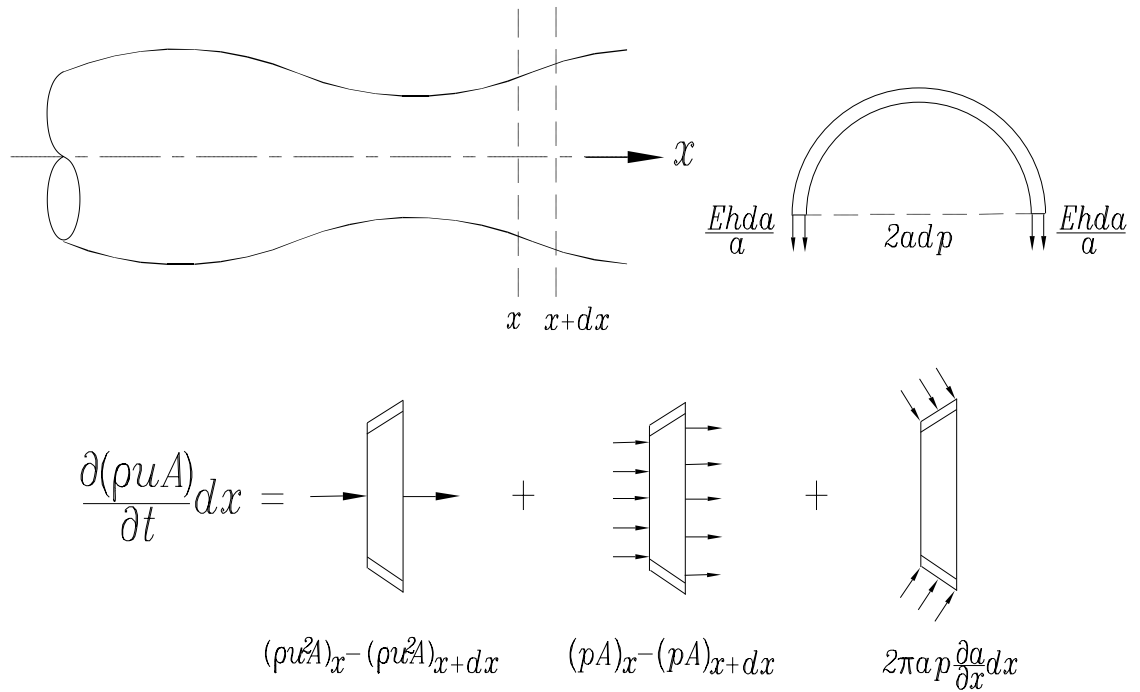


Figure 4: Forces on the artery wall.

Let the pressure outside the artery be constant, say zero. The change in the tube radius must be caused by the change in blood pressure. Referring to figure ??, the elastic strain due to the lengthening of the circumference is  $2\pi da/2\pi a = da/a$ . Let  $h$  be the artery wall thickness, assumed to be much smaller than  $a$ , and Young's modulus  $E$ . The change in elastic force is  $2Ehda/a$  which must be balanced by the changing in pressure force  $2a dp$ , i.e.,

$$\frac{2Eh da}{a} = 2a dp,$$

which implies

$$\frac{dp}{da} = \frac{Eh}{a^2} \quad \text{or} \quad \frac{dp}{dS} = \frac{\sqrt{\pi}Eh}{S^{3/2}} \tag{5.6}$$

Pressure increases with the tube radius, but the rate of increase is smaller for larger radius. Upon integration we get the equation of state

$$p - p_o = -E(h/a) = -\sqrt{\pi}Eh/\sqrt{S} \tag{5.7}$$

Eq (5.5) may now be rewritten as

$$S \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} \right) = -\frac{S}{\rho} \frac{\partial p}{\partial x} = -C^2 \frac{\partial S}{\partial x} \tag{5.8}$$

where  $C$  is defined by

$$C = \sqrt{\frac{S}{\rho} \frac{dp}{dS}} = \sqrt{\frac{Eh}{2\rho a}} \quad (5.9)$$

and has the dimension of velocity. In view of (5.6), equations (5.1) and (5.8) are a pair of nonlinear equations for the two unknowns  $u$  and  $S$ .

For infinitesimal amplitudes we can linearize these equations. Let  $a = a_o + a'$  with  $a' \ll a_o$  then the (5.1) becomes, to the leading order,

$$\frac{\partial a'}{\partial t} + \frac{a_o}{2} \frac{\partial u}{\partial x} = 0 \quad (5.10)$$

The linearized momentum equation is

$$\rho_o \frac{\partial u}{\partial t} = -\frac{\partial p}{\partial x} \quad (5.11)$$

The linearized form of (5.6) is

$$dp = \frac{2Eh}{a_o^2} da' \quad (5.12)$$

which can be used in (5.11) to get

$$\rho \frac{\partial u}{\partial t} = -\frac{Eh}{a_o^2} \frac{\partial a'}{\partial x} \quad (5.13)$$

Finally (5.2) and (5.8) can be combined to give the wave equation:

$$\frac{\partial^2 a'}{\partial t^2} = c_o^2 \frac{\partial^2 a'}{\partial x^2} \quad (5.14)$$

where

$$c_o = \sqrt{\frac{Eh}{2\rho a_o}} \quad (5.15)$$

Alternately one can eliminate  $a$  to get an equation for  $u$

$$\frac{\partial^2 u}{\partial t^2} = c_o^2 \frac{\partial^2 u}{\partial x^2} \quad (5.16)$$

Because of (5.12), the dynamic pressure is governed also by

$$\frac{\partial^2 p}{\partial t^2} = c_o^2 \frac{\partial^2 p}{\partial x^2} \quad (5.17)$$

All unknowns are governed by the same equation due to linearity and the fact that all coefficients are constants.

*Comments on linearization:*

To find out the accuracy of linearization, it is useful to estimate first the scales of motion. Let  $A, T, L, U$  and  $P$  denote the scales of  $a', t, x, u$  and  $p'$  respectively. It is natural to take  $L = c_o T$ . From (5.1) (5.5) and (5.6) we get the relations among the scales of dynamical quantities

$$\frac{a_o A}{T} = \frac{U a_o^2}{L}, \quad \text{hence } U = \frac{A L}{a_o T}$$

$$P = \frac{E h A}{a_o^2}$$

$$\frac{U}{T} = \frac{1}{\rho} \frac{P}{L} = \frac{1}{\rho} \frac{1}{L/T} \frac{E h A}{a_o^2}$$

It follows that

$$\frac{A L}{a_o T} = \frac{1}{\rho} \frac{1}{L/T} \frac{E h A}{a_o^2}$$

hence,

$$\frac{L^2}{T^2} = \frac{E h}{\rho a_o} = c_o^2$$

With these scales the ratio of a typical nonlinear term to a linear term is

$$\frac{u \frac{\partial u}{\partial x}}{\frac{\partial u}{\partial t}} \sim \frac{U^2/L}{U/T} = \frac{U}{L/T} = \frac{A}{a_o}$$

Hence the condition for linearization is that

$$\frac{A}{a_o} \ll 1$$

## 6 Shallow water waves and linearization

### 6.1 Nonlinear governing equations

If water in a lake or along the sea coast is disturbed, waves can be created on the surface, due to the restoring force of gravity. Consider the basic laws governing the motion of long waves in shallow water of constant density and negligible viscosity. Referring to Figure 5, let the  $z$  axis be directed vertically upward and the  $x, y$  plane lie in the initially calm water surface,  $h(x, y)$  denote the depth below the still sea level, and  $\zeta(x, y, t)$  the vertical displacement of the free surface. Take the differential approach again and consider the fluid flow through a vertical column with the base  $dxdy$ .

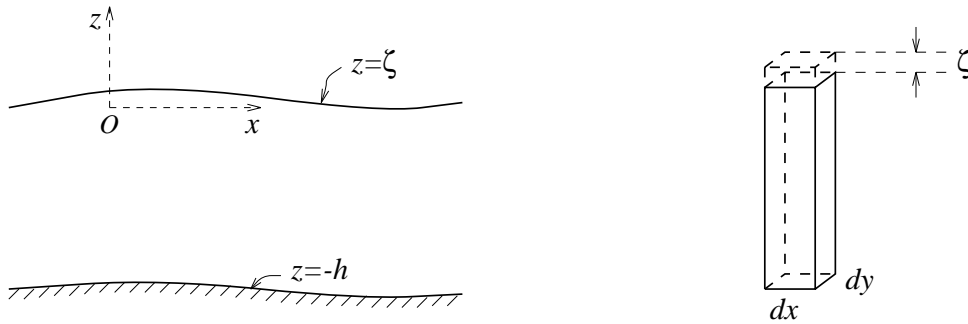


Figure 5: A column element of fluid in a shallow sea

First, the law of mass conservation. The rate of volume increase in the column

$$\frac{\partial \zeta}{\partial t} dx dy$$

must be balanced by the net volume flux into the column from all four vertical sides. In shallow water, the horizontal length scale, characterized by the wavelength  $\lambda$ , is much greater than the vertical length  $h$ . Water flows mainly in the horizontal planes with the velocity  $\mathbf{u}(x, y, t)$ , which is essentially constant in depth. Through the vertical sides normal to the  $x$  axis, the difference between influx through the left and outflux through the right is

$$- [u(\zeta + h)|_{x+dx} - u(\zeta + h)|_x] dy = - \left\{ \frac{\partial}{\partial x} [u(\zeta + h)] + O(dx) \right\} dx dy.$$

Similarly, through the vertical sides normal to the  $y$  axis, the difference between influx through the front and outflux through the back is

$$- [v(\zeta + h)|_{y+dy} - v(\zeta + h)|_y] dx = - \left\{ \frac{\partial}{\partial y} [v(\zeta + h)] + O(dy) \right\} dy dx.$$

Omitting terms of higher order in  $dx, dy$ , we invoke mass conservation to get

$$\frac{\partial \zeta}{\partial t} dxdy = - \left\{ \frac{\partial}{\partial x} [u(\zeta + h)] + \frac{\partial}{\partial y} [v(\zeta + h)] + O(dx, dy) \right\} dxdy.$$

In the limit of vanishing  $dx, dy$ , we have, in vector form,

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot [\mathbf{u}(\zeta + h)] = 0. \quad (6.1)$$

This equation is nonlinear because of the quadratic product of the unknowns  $\mathbf{u}$  and  $\zeta$ .

Now the law of conservation of momentum. In shallow water the vertical momentum balance is dominated by pressure gradient and gravity, which means that the distribution of pressure is hydrostatic:

$$p = \rho g (\zeta - z), \quad (6.2)$$

where the atmospheric pressure on the free surface is ignored. Consider now momentum balance in the  $x$  direction. The net pressure force on two vertical sides normal to the  $x$  direction is

$$\begin{aligned} dxdy \frac{\partial}{\partial x} \int_{-h}^{\zeta} p dz &= -dxdy \frac{\partial}{\partial x} \int_{-h}^{\zeta} \rho g (\zeta - z) dz \\ &= -\rho g (\zeta + h) \frac{\partial (\zeta + h)}{\partial x} dxdy. \end{aligned}$$

The hydrodynamic reaction from the sloping bottom to the fluid is

$$-p \frac{\partial h}{\partial x} dxdy = \rho g (\zeta + h) \frac{\partial h}{\partial x} dxdy.$$

The change of fluid momentum consists of two parts. One part is due to the time rate of momentum change in the water column

$$\left\{ \frac{\partial}{\partial t} [\rho u (\zeta + h)] \right\} dxdy,$$

and the other is due to the net flux of momentum through four vertical sides:

$$\frac{\partial}{\partial x} [\rho u^2 (\zeta + h)] dxdy + \frac{\partial}{\partial y} [\rho uv (\zeta + h)] dydx.$$

Equating the total rate of momentum change to the net pressure force on the sides and on the bottom, we get

$$\begin{aligned} \frac{\partial}{\partial t} [\rho u (\zeta + h)] + \frac{\partial}{\partial x} [\rho u^2 (\zeta + h)] + \frac{\partial}{\partial y} [\rho uv (\zeta + h)] \\ = -g (\zeta + h) \frac{\partial (\zeta + h)}{\partial x} + g (\zeta + h) \frac{\partial h}{\partial x}. \end{aligned}$$

The left-hand side can be simplified to

$$\begin{aligned} \left( \frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \right) (\zeta + h) &+ u \left\{ \frac{\partial \zeta}{\partial t} + \frac{\partial}{\partial x} [u(\zeta + h)] + \frac{\partial}{\partial y} [v(\zeta + h)] \right\} \\ &= \frac{\partial \zeta}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} \end{aligned}$$

by invoking continuity (1.6.1). Hence the  $x$  momentum equation reduces to

$$\frac{\partial u}{\partial t} + u \frac{\partial u}{\partial x} + v \frac{\partial u}{\partial y} = -g \frac{\partial \zeta}{\partial x}. \quad (6.3)$$

Similarly, momentum balance in the  $y$  direction requires

$$\frac{\partial v}{\partial t} + u \frac{\partial v}{\partial x} + v \frac{\partial v}{\partial y} = -g \frac{\partial \zeta}{\partial y}. \quad (6.4)$$

These two equations can be summarized in the vector form:

$$\frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} = -g \nabla \zeta. \quad (6.5)$$

Equations (6.1) and (6.5) are coupled nonlinear partial differential equations for three scalar unknowns  $\mathbf{u}$  and  $\zeta$ .

Now the boundary and initial conditions. On a shoreline  $S$ , there can be no normal flux, therefore,

$$h \mathbf{u} \cdot \mathbf{n} = 0 \quad \text{on } S, \quad (6.6)$$

where  $\mathbf{n}$  denotes the unit normal vector pointing horizontally into the shore. This condition is applicable not only along a cliff shore where  $h$  is finite, but also on a shoreline where  $h = 0$ , as long as the waves are gentle enough not to break. In the latter case the whereabouts of the shoreline is unknown *a priori* and must be found as a part of the solution.

At the initial instant, one may assume that the displacement  $\zeta(x, y, 0)$  and the vertical velocity of the entire free surface  $\frac{\partial}{\partial t} \zeta(x, y, 0)$  is known. These conditions complete the formulation of the nonlinear shallow water wave problem.

## 6.2 Linearization for small amplitude

For small amplitude waves

$$\frac{\zeta}{h} \sim \frac{A}{h} \ll 1, \quad (6.7)$$

where  $A$  is the characteristic amplitude. Equation (1.6.1) may be simplified by neglecting the quadratic term

$$\frac{\partial \zeta}{\partial t} + \nabla \cdot h \mathbf{u} = 0. \quad (6.8)$$

Denoting the time scale by the wave period  $T$  and the horizontal length scale by the wavelength  $\lambda$ , we equate the order of magnitudes of the remaining two terms above to get

$$\frac{A}{T} \sim \frac{uh}{\lambda}, \quad \text{implying} \quad \frac{A}{h} \sim \frac{uT}{\lambda} \ll 1.$$

Now let us estimate the importance of the quadratic term  $\mathbf{u} \cdot \nabla \mathbf{u}$  in the momentum equation by assessing the ratio

$$\frac{\mathbf{u} \cdot \nabla \mathbf{u}}{\frac{\partial \mathbf{u}}{\partial t}} = O\left(\frac{uT}{\lambda}\right) \ll 1.$$

Clearly the quadratic term representing convective inertia can also be ignored in the first approximation, and the momentum equation becomes

$$\frac{\partial \mathbf{u}}{\partial t} = -g \nabla \zeta. \quad (6.9)$$

Both the continuity (1.6.8) and momentum (1.6.9) equations are now *linearized*.

In view of (1.6.9) the boundary condition on the shoreline (1.6.6) can be expressed, instead, as

$$h \frac{\partial \zeta}{\partial n} = 0 \quad \text{on} \quad S. \quad (6.10)$$

Consistent with the linearized approximation, the shoreline position can be prescribed *a priori*.

Equations (6.8) and (6.9) can be combined by the process of cross differentiation. First differentiate (6.8) with respect to  $t$ ,

$$\frac{\partial}{\partial t} \left\{ \frac{\partial \zeta}{\partial t} + \nabla \cdot (h \mathbf{u}) \right\} = 0,$$

then take the divergence of the product of (6.9) and  $h$ ,

$$\nabla \cdot \left\{ h \frac{\partial \mathbf{u}}{\partial t} \right\} = -\nabla \cdot (gh \nabla \zeta).$$

The difference of these two equations gives

$$\frac{\partial^2 \zeta}{\partial t^2} = \nabla \cdot (gh \nabla \zeta). \quad (6.11)$$

For a horizontal bottom  $h = \text{constant}$ ,

$$\frac{1}{c^2} \frac{\partial^2 \zeta}{\partial t^2} = \nabla^2 \zeta, \quad (6.12)$$

where  $c = \sqrt{gh} = O(\lambda/T)$  is the characteristic velocity of infinitesimal wave motion. Equation (1.6.12) is the two-dimensional extension of the wave equation. If, furthermore, all conditions are uniform in the  $y$  direction,  $\partial/\partial y = 0$ , (6.12) reduces to the familiar form

$$\frac{1}{c^2} \frac{\partial^2 \zeta}{\partial t^2} = \frac{\partial^2 \zeta}{\partial x^2}. \quad (6.13)$$

## 7 Sound in fluids

The basic equations governing an inviscid and compressible fluid are as follows. Mass conservation:

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{u}) = 0 \quad (7.14)$$

Momentum conservation:

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -\nabla p \quad (7.15)$$

We must add an equation of state

$$p = p(\rho, S) \quad (7.16)$$

where  $S$  denotes the entropy. When no temperature gradient is imposed externally and the gradient of the flow is not too large, one can ignore thermal diffusion. The fluid motion is then adiabatic; entropy is constant. As a result  $p = p(\rho, S_o)$  depends only on the density. Eq. (7.14) can be written as

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = - \left( \frac{\partial p}{\partial \rho} \right)_S \nabla \rho \quad (7.17)$$

We shall denote

$$C = \sqrt{\left( \frac{\partial p}{\partial \rho} \right)_S} \quad (7.18)$$

so that

$$\rho \left( \frac{\partial \mathbf{u}}{\partial t} + \mathbf{u} \cdot \nabla \mathbf{u} \right) = -C^2 \nabla \rho \quad (7.19)$$



It is easy to check that  $C$  has the dimension of velocity.

From thermodynamics we also have

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \gamma \left(\frac{\partial p}{\partial \rho}\right)_T \quad (7.20)$$

where  $T$  is the temperature and  $\gamma = c_p/c_v =$  ratio of specific heats.

For a perfect gas the equation of state is

$$p = \rho RT \quad (7.21)$$

where  $R$  is the gas constant. Hence for a perfect gas

$$\left(\frac{\partial p}{\partial \rho}\right)_S = \gamma RT \quad (7.22)$$

Liquids are much less compressible. One usually writes the equation of state as

$$d\rho = \left(\frac{\partial \rho}{\partial p}\right)_T dp + \left(\frac{\partial \rho}{\partial T}\right)_p dT \quad (7.23)$$

Denoting

$$\beta = -\frac{1}{\rho} \left(\frac{\partial \rho}{\partial T}\right)_p \quad (7.24)$$

as the coefficient of thermal expansion and

$$\kappa = \frac{1}{\rho} \left(\frac{\partial \rho}{\partial p}\right)_T \quad (7.25)$$

as the coefficient of isothermal compressibility. Usually  $\beta$  is small and  $\kappa$  much smaller.

Under isothermal conditions it is  $\kappa$  that counts.

The simplest limit is the case where the background density  $\rho_o$  and pressure  $p_o$  are uniform, the fluid is at rest and the dynamic perturbations are infinitesimally small. We can write

$$p = p_o + p', \quad \rho = \rho_o + \rho' \quad (7.26)$$

with  $\rho' \ll \rho_o$  and  $p' \ll p_o$ , and linearize the equations to

$$\frac{\partial \rho'}{\partial t} + \rho_o \nabla \cdot \mathbf{u} = 0 \quad (7.27)$$

and

$$\frac{\partial \mathbf{u}}{\partial t} = -\frac{1}{\rho_o} \nabla p' \quad (7.28)$$

Taking the curl of the second, we get

$$\frac{\partial}{\partial t} \nabla \times \mathbf{u} = 0 \quad (7.29)$$

thus the velocity field is irrotational if it is so initially. We can introduce a potential  $\phi$  by

$$\mathbf{u} = \nabla \phi \quad (7.30)$$

It follows from the momentum equation

$$p' = -\rho_o \frac{\partial \phi}{\partial t} \quad (7.31)$$

Using these we get the wave equation.

$$\frac{\partial^2 \phi}{\partial t^2} = c_o^2 \nabla^2 \phi \quad (7.32)$$

where

$$c_o^2 = \left( \frac{\partial p_o}{\partial \rho_o} \right)_s \quad (7.33)$$

## 8 Flexural waves in a beam on an elastic foundation

Referring to Figure (8), we first describe the momentum conservation of a thin beam. Assume the beam to have a horizontal axis when it is not loaded and to have uniform

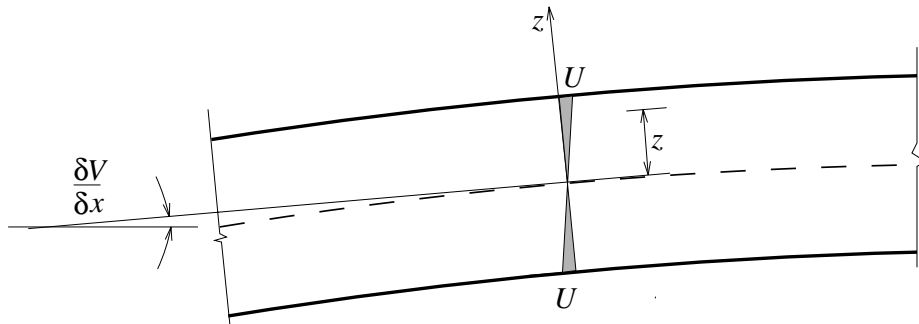


Figure 6: Deflection of a beam

material properties. Let  $V(x)$  denote the upward deflection of the beam axis. If the thickness is small compared to the length and the deflection small compared to the

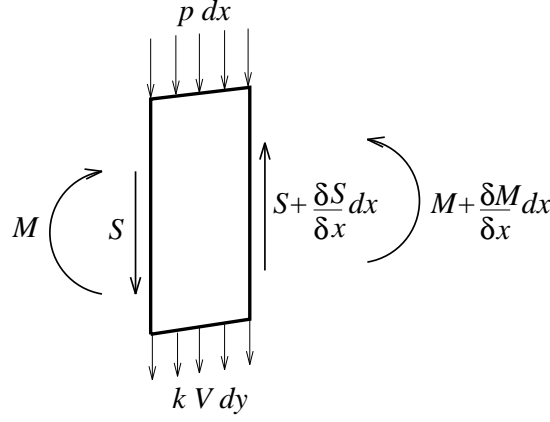


Figure 7: Forces and moments on a beam segment from  $x$  to  $x + dx$ .

thickness, a plane cross section remains approximately plane after deformation. Hence, the longitudinal displacement  $U$  at section  $x$  and height  $z$  above the axis is proportional to  $z$  and to the tilt angle  $\partial V / \partial x$

$$U \cong -z \frac{\partial V}{\partial x}.$$

Consequently, the strain is

$$\varepsilon_x = \frac{\partial U}{\partial x} = -z \frac{\partial^2 V}{\partial x^2}$$

and the longitudinal stress is

$$\sigma_x = E \frac{\partial U}{\partial x} = -Ez \frac{\partial^2 V}{\partial x^2}, \quad (9.1)$$

where  $E$  is Young's modulus. The total moment about the mid-section  $z = 0$  due to the stress distribution across the section is

$$M = - \int_{-h/2}^{h/2} \sigma_x z dz = E \frac{\partial^2 V}{\partial x^2} \int_{-h/2}^{h/2} z^2 dz = EI \frac{\partial^2 V}{\partial x^2}, \quad (9.2)$$

where

$$I = \int_{-h/2}^{h/2} z^2 dz$$

is the moment of inertia of the cross section with respect to its mid-section  $z = 0$ . Consider a length element of the beam from  $x$  to  $x + dx$ , as sketched in Fig. (7). The balance of angular momentum about the center of the element requires that

$$M + \frac{\partial M}{\partial x} dx - M + \left( S + \frac{\partial S}{\partial x} dx \right) \frac{dx}{2} + S \frac{dx}{2} = \rho J dx \frac{\partial^3 V}{\partial x \partial t^2},$$

where  $\rho$  is the mass and  $\rho J$  is the rotatory moment of inertia per unit length of the beam. For a beam with rectangular cross section,  $J = h^2/12$ , where  $h$  is the height of the beam. Thus,

$$S = -\frac{\partial M}{\partial x} + \rho J \frac{\partial^3 V}{\partial x \partial t^2} = -EI \frac{\partial^3 V}{\partial x^3} + \rho J \frac{\partial^3 V}{\partial x \partial t^2}. \quad (9.3)$$

On the other hand, balance of vertical forces requires that

$$S + \frac{\partial S}{\partial x} dx - S = p dx + kV dx + \rho \frac{\partial^2 V}{\partial t^2},$$

where  $k$  is the elastic constant of the lateral support, and  $p(x, t)$  is the distributed load. Making use of (6.5.3), we get

$$\rho \left( 1 - J \frac{\partial^2}{\partial x^2} \right) \frac{\partial^2 V}{\partial t^2} + EI \frac{\partial^4 V}{\partial x^4} + kV = -p(x, t), \quad (9.4)$$

which is a fourth-order partial differential equation, derived first by Lord Rayleigh. If the wavelength  $L$  is much greater than the beam height  $h/L \ll 1$ , then the term representing rotatory inertia is negligible and (6.5.4) may be simplified to

$$\rho \frac{\partial^2 V}{\partial t^2} + EI \frac{\partial^4 V}{\partial x^4} + kV = -p(x, t). \quad (9.5)$$

Now the boundary conditions. For a beam of finite length, each end can be free, clamped, or supported on a hinge. At a free end, there is neither torque nor shear

$$\frac{\partial^2 V}{\partial x^2} = 0, \quad \frac{\partial^3 V}{\partial x^3} = 0. \quad (9.6)$$

At a clamped end, the deflection and slope must vanish

$$V = 0, \quad \frac{\partial V}{\partial x} = 0. \quad (9.7)$$

At a hinged end both the deflection and the torque are zero

$$V = 0, \quad \frac{\partial^2 V}{\partial x^2} = 0. \quad (9.8)$$

For an infinitely long beam, the boundary conditions at infinity depend on the loading. For a transient loading with finite duration,  $V$  should vanish at infinities. For time-harmonic loadings, the disturbance should at most be outgoing waves.

## 10 Homework No.1

1. A membrane is kept taut over an area  $S$ . The membrane has a constant density  $\rho$  per unit area and is under uniform tension in all directions. Derive the governing equation for the lateral displacement  $u(x, y, t)$  of the membrane vibrating under distributed loading of  $p(x, t)$  per unit area.

2. Consider the longitudinal vibration of a cylindrical rod with one end at  $x = 0$  fixed and the other end at  $x = L$  attached to a mass  $M$ . Before  $t = 0$  the rod is compressed by the length  $\epsilon L$  with  $\epsilon \ll 1$ . At  $t = 0$  the compression is released. State the governing equation and all boundary and initial conditions.

3. Consider the torsional vibration of a cylindrical rod of circular cross section of radius  $a$ . Let  $\theta(x, t)$  = angular displacement of the cross section at  $x$ ,  $d\sigma$  = area element in the cross section and located at the distance  $r$  from the axis, see figure 8. Let  $\tau$  be the shear stress,  $G$  the shear modulus of elasticity, and  $\phi$  the angular displacement of a line originally parallel to the axis. Show that

$$\phi = r \frac{\partial \theta}{\partial x} \quad (10.1)$$

Invoke Hooke's law  $\tau = G\phi$  and show that the total torque applied to the cross section at  $x$  is

$$M = G \frac{\partial \theta}{\partial x} \iint_S r^2 d\sigma = GJ \frac{\partial \theta}{\partial x} \quad (10.2)$$

where

$$J = \iint_S r^2 d\sigma \quad (10.3)$$

is the polar moment of inertia of the cross section. Let  $I$  be the moment of inertia per unit length of the rod. Show that

$$\frac{\partial^2 \theta}{\partial t^2} = \frac{GJ}{I} \frac{\partial^2 \theta}{\partial x^2} \quad (10.4)$$

4. During an earthquake, water in a reservoir exerts hydrodynamic pressure on a dam that may fail. Formulate the dam-reservoir interaction problem under the following idealizations. The reservoir is infinitely long and has a uniform rectangular cross section.

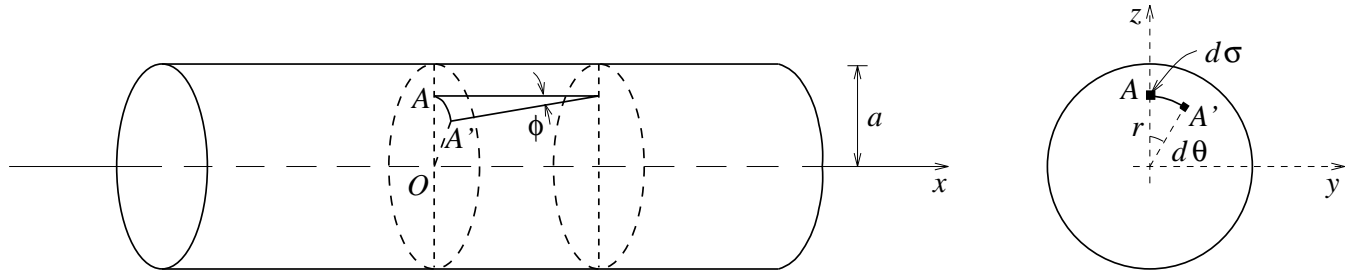


Figure 8: Torsion of a circular cylinder

Water is present only on one side of the dam ( $x > 0$ ) and has the constant depth  $h$ . Before  $t = 0$ , all is calm. After  $t = 0$  the dam is forced to vibrate horizontally so that

$$u(0, y, z, t) = \begin{cases} u_o(y, z, t) = \text{prescribed}, & 0 < t < T, \\ 0, & t > T. \end{cases} \quad (10.5)$$

The free surface is exposed to constant atmospheric pressure. The reservoir bottom is rigid and does not vibrate vertically (!!!). Neglect gravity but consider compressibility of water because of the high frequency ( $\sim O(100)\text{Hz}$ ). Express all governing equations including the boundary conditions in terms of the velocity potential  $\phi$  defined by  $(u, v, w) = \nabla\phi$ .

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