MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

2.04A Systems and Controls
Spring 2013

Practice Problems #1

Please do not turn in – practice only Posted: Thursday, Feb. 21, '13

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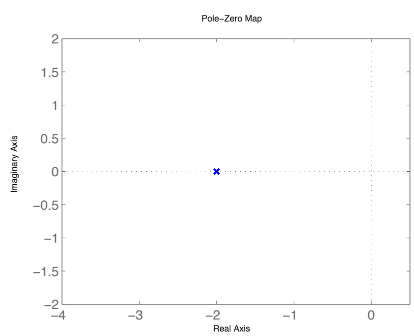
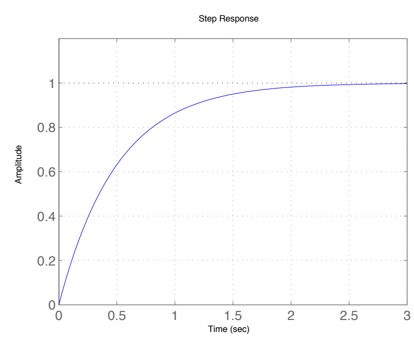
1. For each one of the following transfer functions, identify the zeros and poles, show their locations on the complex (s–) plane, then and derive and plot the step response.

1.a)

$$T(s) = \frac{2}{s+2}$$

<u>Answer:</u> Pole: p=-2, Zero: none



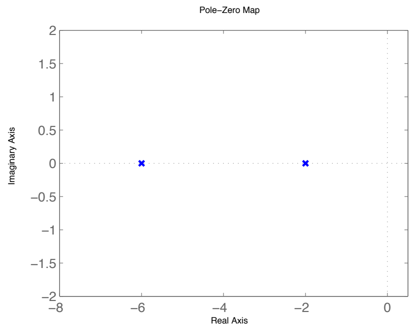


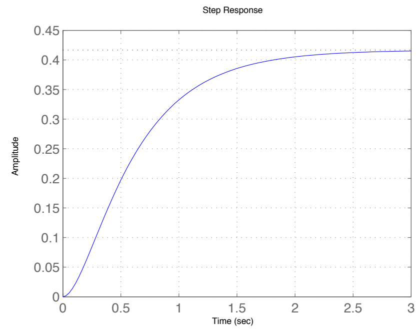
Step response $(1 - e^{-2t})u(t)$. 1st order system.

1.b)

$$T(s) = \frac{5}{(s+2)(s+6)}$$

<u>Answer:</u> Poles: $p_1 = -2$, $p_2 = -6$, Zero: none





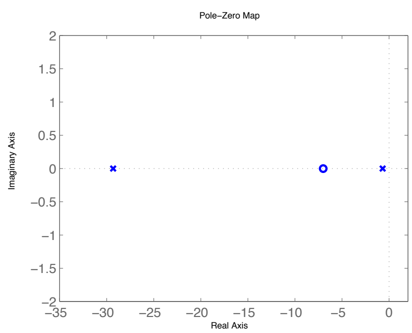
$\omega_n^2 = 12$ and $\zeta = 4/\sqrt{12} = 1.15 > 1$. 2nd order overdamped system.

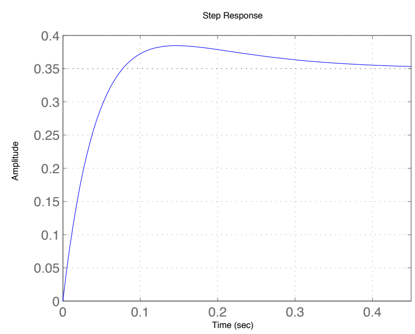
Step response $\left[1 + K_1 e^{-p_1 t} + K_2 e^{-p_2 t}\right] u(t)$.

1.c)

$$T(s) = \frac{10(s+7)}{(s+10)(s+20)}$$

<u>Answer:</u> Poles: $p_1 = -10$, $p_2 = -20$, Zeros: $z_1 = -7$





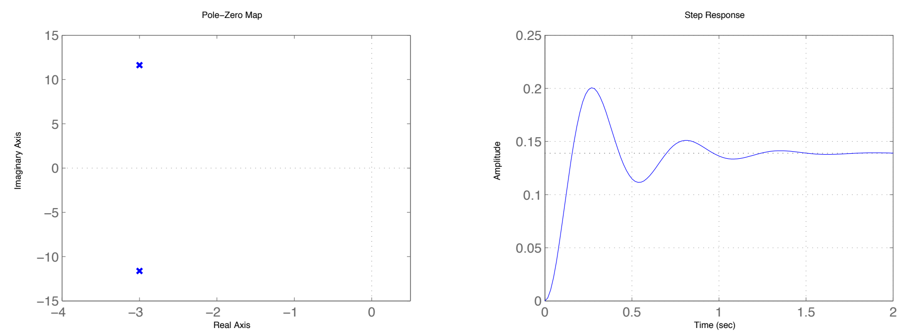
$\omega_n^2 = 200$ and $\zeta = 30/2/\sqrt{200} = 1.06 > 1$. 2nd order overdamped system.

Step response $\left[1 + K_1 e^{-p_1 t} + K_2 e^{-p_2 t}\right] u(t)$.

1.d)

$$T(s) = \frac{20}{s^2 + 6s + 144}$$

<u>Answer:</u> Poles: $p_1 = -3 + j11.619$, $p_2 = -3 - j11.619$, Zeros: none

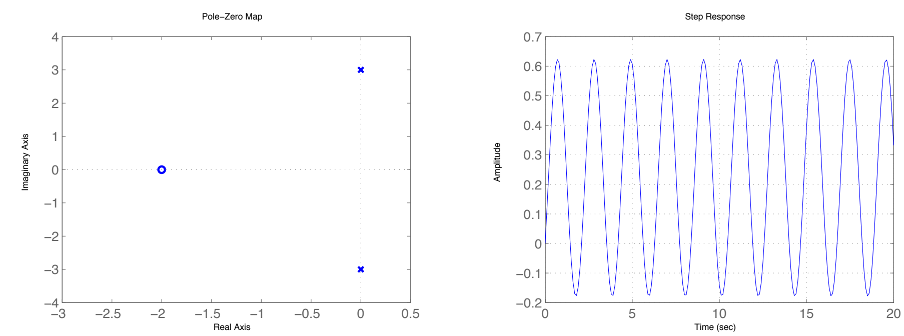


$\omega_n^2 = 144$ and $\zeta = 6/2/\sqrt{144} = 0.25 < 1$. 2nd order underdamped system.
Step response $\left[1 - Ae^{-\sigma_d t}\cos\left(\omega_d t - \phi\right)\right]u(t)$.

1.e)

$$T(s) = \frac{s+2}{s^2+9}$$

<u>Answer:</u> Poles: $p_1 = 3j,\ p_2 = -3j,$ Zero: $z = -2$

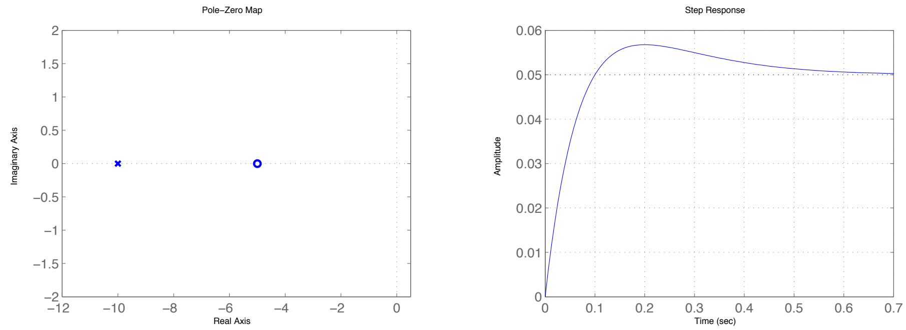


$\omega_n^2 = 9$ and $\zeta = 0$. 2nd order undamped system.
Step response $\left[1 - K_1\sin\left(3t\right) + K_2\cos\left(3t\right)\right]u(t)$.

1.f)

$$T(s) = \frac{(s+5)}{(s+10)^2}$$

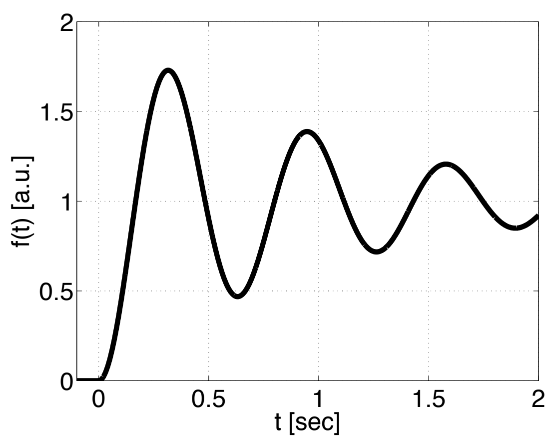
<u>Answer:</u> Poles: $p = -10(double),$ Zeors: $z = -5$



$\omega_n^2 = 100$ and $\zeta = 1$. 2nd order critically damped system.

Step response $[K_0 + K_1 e^{-10t} - K_2 t e^{-10t}]\, u(t)$.

2. A second–order system has the step response shown below.[1] Determine its transfer function.



Answer: This is under–damped 2nd order system. Starting from the transfer function of the second order system

$$A\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega^2},$$

we have to decide the parameters of A(constant),ζ(damping ratio) and ω_n(natural frequency).

From the final value theorem,

$$\lim_{s\to 0} \frac{1}{s} s \frac{A\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = A$$

[1]a.u. denotes arbitrary units; its use appropriate when we consider a function that does not correspond to any particular physical quantity.

and the steady state value is 1 (from the given figure). Therefore, $A = 1$.

The step response of the under–damped second order system is

$$\left[1 - ae^{-\sigma_d t}\cos\left(\omega_d t - \phi\right)\right] u(t),$$

where $\sigma_d = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

From the lecture note 7 (pp. 26), %OS $= \exp\left(-\frac{\zeta\pi}{1-\zeta^2}\right)$: 72%.

Thus the damping ratio $\zeta \approx 0.1$.

To get the natural frequency, we choose two peak points at $t_1 = 0.35$ sec and $t_2 = 0.95$ sec. The cosine term will be 1 at the peaks, so that we can consider exponential decay term only.

$$\begin{aligned} f(t_1) &= 1 - ae^{-\sigma_d t_1} = 1.72 \\ f(t_2) &= 1 - ae^{-\sigma_d t_2} = 1.4 \end{aligned}$$

Dividing the two equations, we obtain

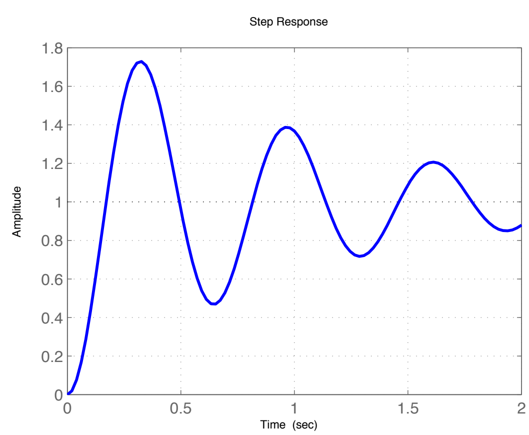
$$\frac{ae^{-\sigma_d t_1}}{ae^{-\sigma_d t_1}} = \frac{1-1.72}{1-1.4}.$$

From that $\sigma_d = \left\{\ln\left(\frac{0.72}{0.4}\right)\right\} / \left\{t_2 - t_1\right\} = 0.9796$. Therefore $\omega_n \approx 9.8$. (The reason why I picked two points instead of one point is to cancel the constant a).

The transfer function is

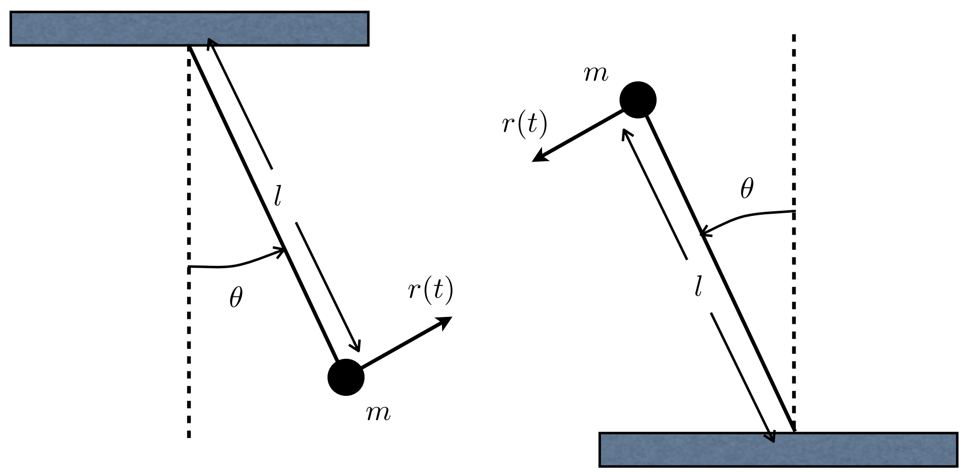
$$\frac{64}{s^2 + 1.96s + 96},$$

and its step response by MATLAB is

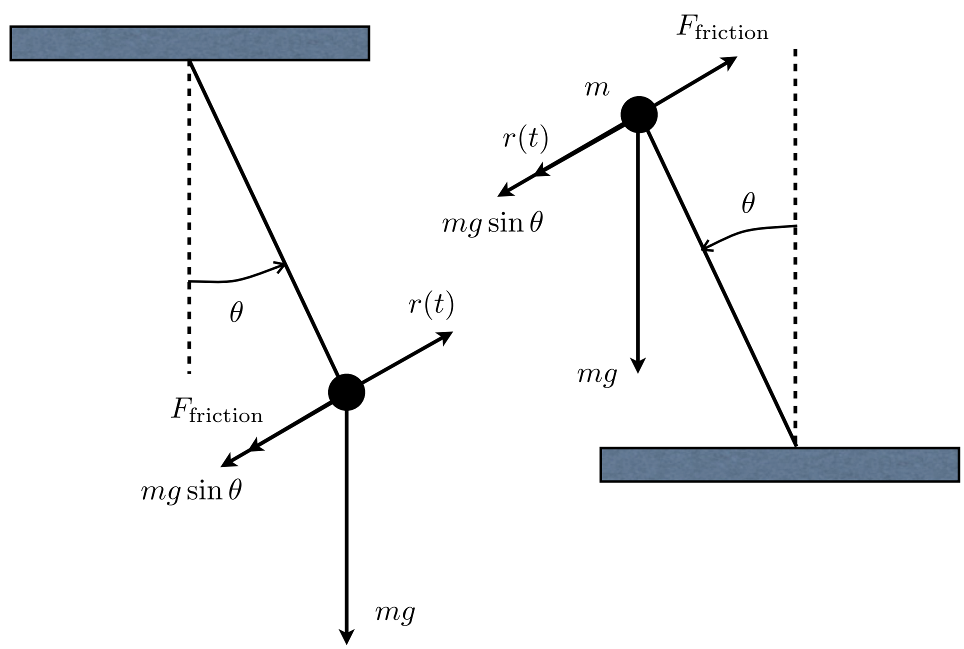


Note that the estimated parameters might be slightly different than the original because our reading of the plot can never be completely accurate.

3. Consider a pendulum and inverted pendulum, as shown in the figure below. In this problem we will explore how they are different physically, and how the difference is mapped out in the Laplace domain. We assume that the mass m and length l are the same in both cases, and that they are both subject to viscous friction with coefficient b from the surrounding medium (in reality, the pendulums would be subject to drag forces as well, but we neglect them here to simplify the problem.) In both cases, the pendulums are driven by a force $r(t)$ applied in the direction tangential to the motion. The initial conditions are $\theta(0) = 0$ and $\dot{\theta}(0) = 0$.



3.a) Derive the equations of motion for the two cases, using the angle $\theta(t)$ as the output variable, and assuming small motion $|\theta| \ll 1$ away from the vertical.



Answer: We begin with the free–body diagrams (FBDs) for the two cases, shown respectively above as well. For the standard pendulum (left–hand side) torque balance from the FBD in conjunction with Newton's law of motion yield

$$ml^2\ddot{\theta}(t) = r(t)l - mgl\sin\theta(t) - F_{\text{friction}}l. \tag{1}$$

Substituting for viscous friction $F_{\text{friction}} = b\dot{\theta}(t)$, and dividing across by ml^2 we obtain the equation of motion

$$\ddot{\theta}(t) + \frac{b}{ml}\dot{\theta}(t) + \frac{g}{l}\sin\theta(t) = \frac{r(t)}{ml}. \tag{2}$$

Assuming that the pendulum never strays too far away from the vertical, $|\theta| \ll 1$, therefore $\sin\theta \approx 1$ and the linearized equation of motion becomes

$$\ddot{\theta}(t) + \frac{b}{ml}\dot{\theta}(t) + \frac{g}{l}\theta(t) = \frac{r(t)}{ml}. \tag{3}$$

Following similar procedure in the case of the inverted pendulum (right–hand side in the above figure) torque balance yields

$$ml^2\ddot{\theta}(t) = r(t) + mgl\sin\theta(t) - F_{\text{friction}}l. \tag{4}$$

from which we obtain the liberalized equation of motion as

$$\ddot{\theta}(t) + \frac{b}{ml}\dot{\theta}(t) - \frac{g}{l}\theta(t) = \frac{r(t)}{ml}. \tag{5}$$

Comparing (5) and (3), we can see that there is only a difference in sign compared to the standard pendulum case. However, we will see that this difference has a profound effect on the system behavior.

3.b) Derive the transfer function $\Theta(s)/R(s)$ for the two cases, and draw the locations of any poles and zeros that you find on the complex (s–) plane.

Answer: Starting with the standard pendulum, Laplace transforming both sides of (3) and taking into account the zero initial conditions, we obtain

$$\frac{\Theta(s)}{R(s)} = \frac{\dfrac{1}{ml}}{s^2 + \dfrac{b}{ml}s + \dfrac{g}{l}}. \tag{6}$$

From this we conclude that there are no zeros in the transfer function, and there are two poles located at

$$s_{p\pm} = \frac{1}{2}\left[-\frac{b}{ml} \pm \sqrt{\left(\frac{b}{ml}\right)^2 - 4\frac{g}{l}}\right]. \tag{7}$$

Under normal conditions, we expect friction to be weak, certainly so that the friction coefficient satisfies $b < 2m\sqrt{gl}$. Then the two poles become complex, with real part

$$-\frac{b}{2ml}$$

and conjugate imaginary parts

$$\pm\sqrt{\left(4\frac{g}{l} - \frac{b}{ml}\right)^2}.$$

Matching the coefficients of the polynomial in the denominator with those of our "standard" 2^{nd}–order system transfer function, we find that for the standard pendulum

$$\omega_{\mathrm{n}} = 2\sqrt{\frac{g}{l}}, \qquad \zeta = \frac{1}{2}\frac{b}{m\sqrt{gl}}. \tag{8}$$

In the case of the inverted pendulum, similar procedure yields

$$\frac{\Theta(s)}{R(s)} = \frac{\frac{1}{ml^2}}{s^2 + \frac{b}{ml}s - \frac{g}{l}}. \tag{9}$$

Again, there are no zeros, and there are two poles located at

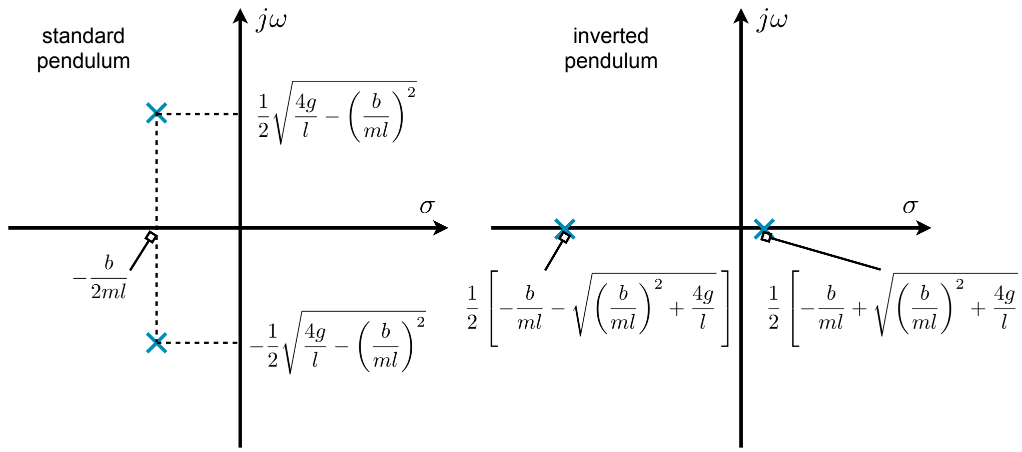
$$s_{p\pm} = \frac{1}{2}\left[-\frac{b}{ml} \pm \sqrt{\left(\frac{b}{ml}\right)^2 + 4\frac{g}{l}}\right]. \tag{10}$$

Unlike the previous case, both poles are now real, and, moreover, the pole located at

$$s_{p+} = \frac{1}{2}\left[-\frac{b}{ml} + \sqrt{\left(\frac{b}{ml}\right)^2 + 4\frac{g}{l}}\right].$$

is positive! This indicates that the impulse, step, etc. responses of the inverted pendulum contain exponentially increasing terms. (You can easily verify the effect of the exponentially increasing term by trying to hold a pen upright on your palm.) It is important to note that, since $\theta(t)$ may increase exponentially in this case, the assumption of small $|\theta| \ll 1$ will break down after some time t, and then system behavior won't be well modeled by our equations.

The locations of the system poles for the standard and inverted pendulum on the complex plane are shown on the left– and right–hand side diagrams below, respectively, on the next page. Note the locations of the poles for the underdamped standard pendulum are symmetric with respect to the real axis (*i.e.*, complex conjugates). Also note the location of one pole of the inverted pendulum on the right–hand side of the complex plane. Systems with at least one pole on the right–hand side of the complex plane are called *unstable.*



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