2.035: Selected Topics in Mathematics with Applications
Final Exam – Spring 2007

Every problem in the calculus of variations has a solution,
provided the word "solution" is suitably understood.
David Hilbert (1862-1943)

Work any 5 problems.

Pick-up exam: 12:30 PM on Tuesday May 8, 2007

Turn-in solutions: 11:00 AM on Tuesday May 15, 2007

You may use notes in your own handwriting (taken during and/or after class) and all handouts (including anything I emailed to you) and my bound notes. Do not use any other sources.

Do not spend more than 2 hours on any one problem.

Please include, on the first page of your solutions, a signed statement confirming that you adhered to all of the instruction above.

Problem 1: Using first principles (i.e. don't use some formula like $d/dx(\partial F/\partial \phi') - \partial F/\partial \phi = 0$ but rather go through the steps of calculating δF and simplifying it etc.) determine the function $\phi \in \mathsf{A}$ that minimizes the functional

$$F\{\phi\} = \int_0^1 \left(\frac{1}{2}(\phi')^2 + \phi\phi'' + \phi\right) dx$$

over the set

$$\mathsf{A} = \{\phi | \phi : [0,1] \to \mathbb{R},\ \phi \in C^2[0,1]\}.$$

Note that the values of ϕ are not specified at either end.

Problem 2: A problem of some importance involves navigation through a network of sensors. Suppose that the sensors are located at fixed positions and that one wishes to navigate in such a way that the navigating observer has minimal exposure to the sensors. Consider the following simple case of such a problem. See figure on last page.

A single sensor is located at the origin of the x, y-plane and one wishes to navigate from the point $A = (a, 0)$ to the point $B = (b\cos\beta, b\sin\beta)$. Let $(x(t), y(t))$ denote the location of the observer at time t so that the travel path is described parametrically by $x = x(t), y = y(t), 0 \le t \le T$. The exposure of the observer to the sensor is characterized by

$$E\{x(t), y(t)\} = \int_0^T I(x(t), y(t))\, v(t)\ dt$$

where $v(t)$ is the speed of the observer and the "sensitivity" I is given by

$$I(x, y) = \frac{1}{\sqrt{x^2 + y^2}}\ .$$

Determine the path from A to B that minimizes the exposure.

Hint: Work in polar coordinates $r(t), \theta(t)$ and find the path in the form $r = r(\theta)$.
Remark: For further background on this problem including generalization to n sensors, see the paper Qingfeng Huang, "Solving an Open Sensor Exposure Problem using Variational Calculus", Technical Report WUCS-03-1, Washington University, Department of Computer Science and Engineering, St. Louis, Missouri, 2003.

Problem 3: Consider a domain $\mathcal{D}$ of the x, y-plane whose boundary $\partial\mathcal{D}$ is smooth. Let A denote the set of all functions $\phi(x, y)$ that are defined and suitably smooth on $\mathcal{D}$ and which vanish on the boundary of $\mathcal{D}$: $\phi = 0$ for $(x, y) \in \partial\mathcal{D}$. Define the functional

$$F\{\phi\} = \int_{\mathcal{D}} \left(\left(\frac{1}{2}\frac{\partial^2\phi}{\partial x \partial y}\right)^2 + \frac{1}{2}\phi^2 + q\phi\right) dA$$

for all $\phi \in \mathsf{A}$ where $q = q(x,y)$ is a given function on $\mathcal{D}$. You are asked to minimize the functional F over the set A. Determine the boundary-value problem that the minimizer must satisfy. (You do NOT need to solve it.)

Problem 4: Derive the two corner conditions

$$\left.\frac{\partial f}{\partial \phi'}\right|_{x=s-} = \left.\frac{\partial f}{\partial \phi'}\right|_{x=s+}, \qquad \left.\left(f - \phi'\frac{\partial f}{\partial \phi'}\right)\right|_{x=s-} = \left.\left(f - \phi'\frac{\partial f}{\partial \phi'}\right)\right|_{x=s+},$$

that a minimizer of the functional

$$F\{\phi\} = \int_0^1 f(x, \phi, \phi')\, dx$$

must satisfy if the minimizer is continuous and has piecewise continuous derivatives; here $x = s$ denotes the location of a discontinuity in ϕ' and the value of s is not known a priori. *Remark:* This question simply asks you to derive the results that I quoted in class without proof on Thursday May 3.

Problem 5: Consider the eigenvalue problem (1), (2) for finding the eigenfunctions $\phi(x)$ and eigenvalues λ of the boundary-value problem

$$a\,\phi''(x) - b\,\phi(x) + \lambda c\,\phi(x) = 0 \qquad \text{for} \quad 0 \le x \le 1, \tag{1}$$

$$\phi(0) = 0, \quad \phi(1) = 0, \tag{2}$$

where a, b, c and λ are *constants*. (Remark: There is NO typo in the equations above, just in case you expected it to be $b\phi'(x)$ instead of $b\phi(x)$.)

a) Show that minimizing the functional

$$F\{\phi\} = \int_0^1 \left[a\,(\phi')^2 + (b - \lambda c)\,(\phi)^2\right] dx$$

over the set $\mathsf{A} = \{\phi | \phi : [0,1] \to \mathbb{R},\ \phi \in C^2[0,1],\ \phi(0) = \phi(1) = 0\}$ leads to equation (1) as its Euler equation.

b) Since the functional F involves the unknown eigenvalues λ, the preceding is not a particularly useful variational statement, which motivates us to look for other variational formulations of the eigenvalue problem. Show that minimizing the functional

$$G\{\phi\} = \int_0^1 \left[a\,(\phi')^2 + b\,(\phi)^2\right] dx$$

over the set A *subject to the constraint*

$$H\{\phi\} = \int_0^1 c\,(\phi)^2\,dx = \textsf{constant}$$

leads to equation (1) as its Euler equation with λ as a Lagrange multiplier.

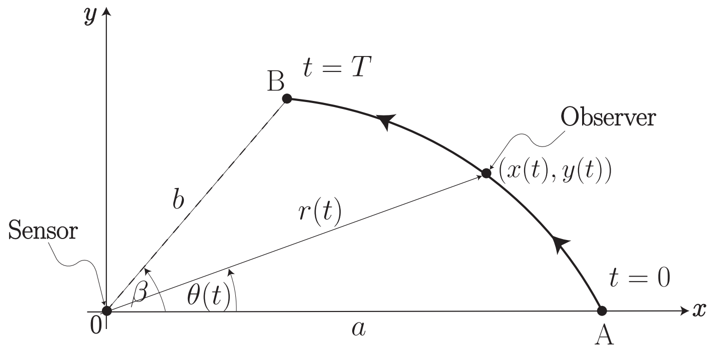
c) Show that minimizing the functional

$$I\{\phi\} = \frac{\int_0^1 \left[a\,(\phi')^2 + b\,(\phi)^2\right] dx}{\int_0^1 c\,\phi^2\,dx}$$

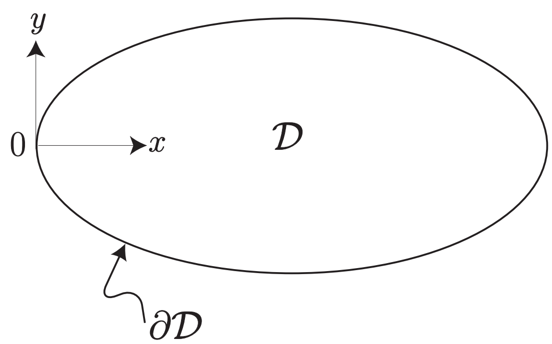
over the set A leads to equation (1) as its Euler equation, and that the values of I at the minimizers are the eigenvalues.

Problem 6: Suppose that an airplane flies in the x, y-plane at a constant speed v_o relative to the wind. The flight path begins and ends at the same location so that the path is a loop. The flight takes a given duration T. Assume that the wind velocity has a constant magnitude $c\,(< v_o)$ and a constant direction (which, without loss of generality, you can take to be the x-direction).

Along what closed curve should the plane fly if the flight path is to enclose the greatest area?



Problem 2: An observer moves along a path in the x, y-plane such that its rectangular cartesian coordinates at time t are $(x(t), y(t))$ (or its polar coordinates are $(r(t), \theta(t))$. A sensor is located at the origin.



Problem 3: A domain $\mathcal{D}$ in the x, y-plane with a smooth boundary $\partial\mathcal{D}$.