2.011 Motion of the Upper Ocean

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Equations of Motion

- Impart an impulsive force on the surface of the fluid to set it in motion (no other forces act on the fluid, except Coriolis).
- Then, for a parcel of water moving with zero friction:

$$\frac{du}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + 2\Omega v \sin\varphi$$

$$\frac{dv}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial y} - 2\Omega u \sin\varphi$$

$$\frac{dw}{dt} = -\frac{1}{\rho}\frac{\partial p}{\partial z} + 2\Omega u \cos\varphi - g$$

$\Omega = 2\,\pi/(\text{sidereal day}) = 7.292 \times 10^{-5}\ \text{rad/s}$

φ is latitude

Simplify the equations:

- Assuming only Coriolis force is acting on the fluid then there will be no horizontal pressure gradients:

$$\frac{\partial p}{\partial x} = \frac{\partial p}{\partial y} = 0$$

- Assuming then the flow is only horizontal (*w*=*0*):

$$\frac{du}{dt} = 2\Omega\, v \sin\varphi = f v$$
$$\frac{dv}{dt} = -2\Omega\, u \sin\varphi = -f u$$

- Coriolis Parameter: $f = 2\,\Omega\,\sin\varphi$

Solve these equations

- Combine to solve for *u*:

$$\frac{du}{dt} = 2\Omega\, v \sin\varphi = f v$$
$$\frac{dv}{dt} = -2\Omega\, u \sin\varphi = -f u$$

$$\frac{du}{dt} = -\frac{1}{f}\frac{d^2v}{dt^2} = f v$$

- Standard Diff. Eq. $\frac{d^2v}{dt^2} + f^2 v = 0$

- Inertial Current Solution:

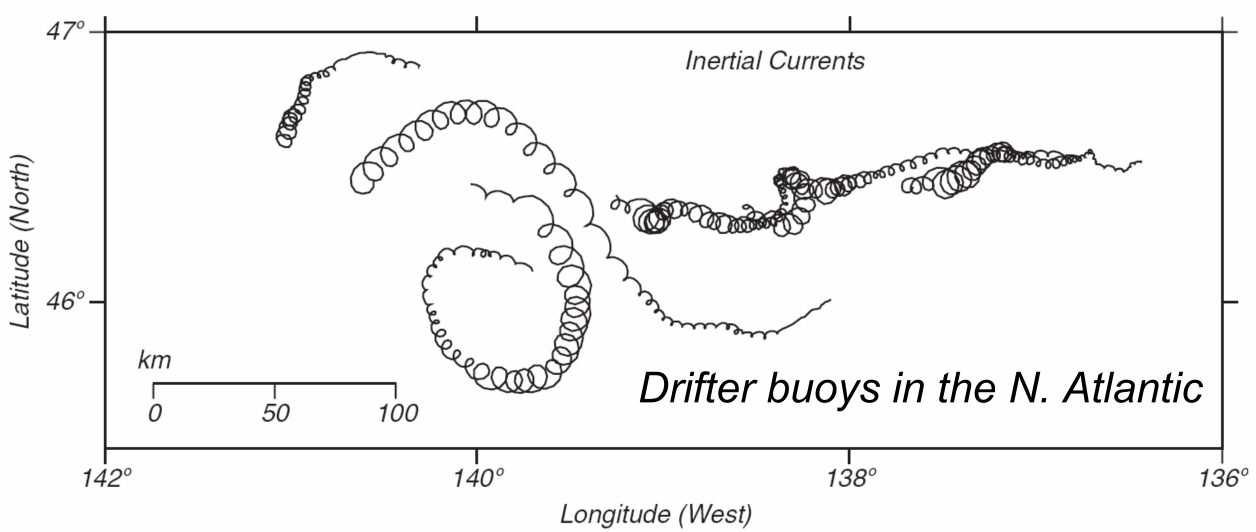
(Inertial Oscillations)

$$u = V \sin ft$$
$$v = V \cos ft$$
$$V^2 = u^2 + v^2$$

Inertial Current

$$u = V \sin ft$$
$$v = V \cos ft$$
$$V^2 = u^2 + v^2$$

- Note this solution are equations for a circle

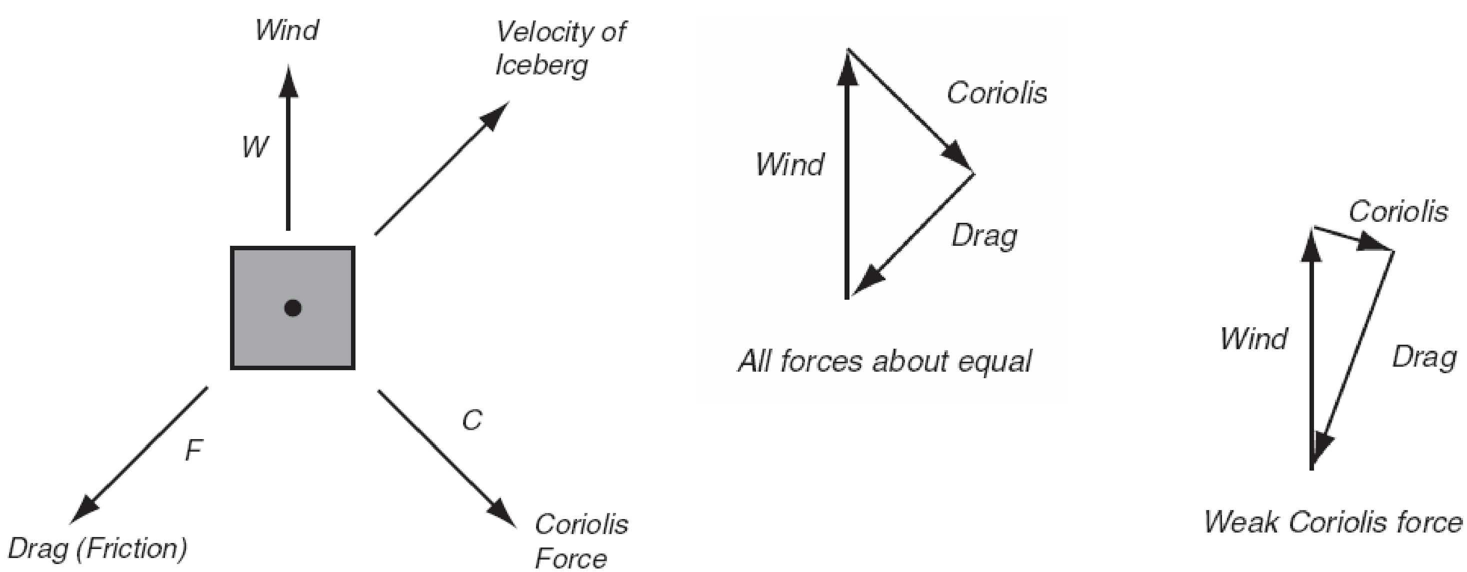


- Circle Diameter: $D_i = 2V/f$
- Inertial Period: $T_i = (2\pi)/f = T_{sd}/(2\sin\varphi)$
- Anti-cyclonic (clockwise) in N. Hemisphere; cyclonic (counterclockwise) in S. Hemi
- Most common currents in the ocean!

Source: Introduction to Physical Oceanography, http://oceanworld.tamu.edu/home/course_book.htm

Ekman Layer

- Steady winds on the surface generate a thin, horizontal boundary layer (i.e. Ekman Layer)
- Thin = O (100 meters) thick
- First noticed by Nansen that wind tended to blow ice at 20-40° angles to the right of the wind in the Arctic.



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Ekman's Solution

- Steady, homogeneous, horizontal flow with friction on a rotating earth
- All horizontal and temporal derivatives are zero

$$\frac{\partial}{\partial t} = \frac{\partial}{\partial x} = \frac{\partial}{\partial y} = 0$$

- Wind Stress in horizontal (x, y) directions

$$T_{xz} = \rho A_z \frac{\partial u}{\partial z}, \qquad T_{yz} = \rho A_z \frac{\partial v}{\partial z}$$

- A_z is an eddy viscosity or diffusivity that replaces kinematic viscosity

Equations of motion with wind stress

$$\rho f\, v + \frac{\partial T_{xz}}{\partial z} = 0$$

$$\rho f\, u - \frac{\partial T_{yz}}{\partial z} = 0$$

Steady, homogeneous, horizontal, viscous, turbulent flow equations for momentum from NSE

$$T_{xz} = \rho\, A_z \frac{\partial u}{\partial z}\,, \qquad T_{yz} = \rho\, A_z \frac{\partial v}{\partial z}$$

$$f v + A_z \frac{\partial^2 u}{\partial z^2} = 0$$

$$-f u + A_z \frac{\partial^2 v}{\partial z^2} = 0$$

$$u = V_0 \exp(az) \cos(\pi/4 + az)$$

$$v = V_0 \exp(az) \sin(\pi/4 + az)$$

Ekman Current

$$u = V_0 \, \exp(az) \, \cos(\pi/4 + az)$$
$$v = V_0 \, \exp(az) \, \sin(\pi/4 + az)$$

- When wind blows north $T = T_{yz}$ *and*

$$V_0 = \frac{T}{\sqrt{\rho_w^2 \, f \, A_z}} \qquad \text{and} \qquad a = \sqrt{\frac{f}{2A_z}}$$

- *At z = 0*

$$u(0) = V_0 \, \cos(\pi/4)$$
$$v(0) = V_0 \, \sin(\pi/4)$$

- Wind stress:

$$T_{yz} = T = \rho_{air} \, C_D \, U_{10}^2$$

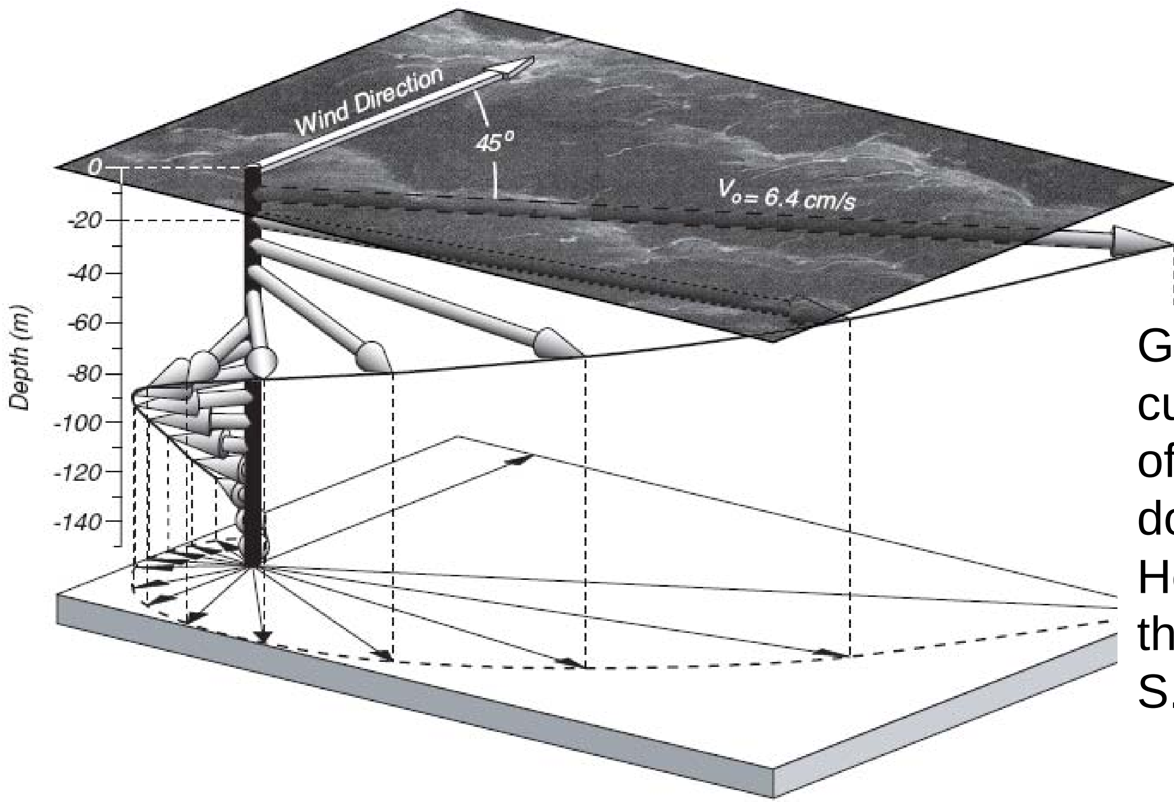
- Surface current speed versus wind speed:

$$V_0 = \frac{0.0127}{\sqrt{\sin|\varphi|}} \, U_{10}, \qquad \text{latitude } |\varphi| \geq 10$$

Ekman Spiral

- Current moves at speed Vo to the north east
- Below the surface the velocity decays exponentially with depth

$$\left[u^2(z) + v^2(z)\right]^{1/2} = V_0 \exp(az)$$



Generally the surface current is 45° to the right of the wind when looking down in the N. Hemisphere and 40° to the left of the wind in the S. Hemi.

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Ekman Layer

- Ekman Layer depth $D_E = \sqrt{\frac{2\pi^2 A_z}{f}} = \frac{7.6}{\sqrt{\sin|\varphi|}} U_{10}$

  for $\rho = 1027\ \mathrm{kg/m^3}$, $\rho_{air} = 1.25\ \mathrm{kg/m^3}$, $C_D = 2.6 \times 10^{-3}$
- Depth below surface that the current is directly opposite surface current: $D_E = \pi/a$

Typical Ekman Depths

	Latitude	
U_{10} [m/s]	15°	45°
5	75 m	45 m
10	150 m	90 m
20	300 m	180 m

Ekman Number

Relates relative magnitude of Coriolis and friction forces

$$E_z = \frac{\text{Friction Force}}{\text{Coriolis Force}} = \frac{A_z \dfrac{\partial^2 u}{\partial z^2}}{f u} = \frac{A_z \dfrac{u}{d^2}}{f u}$$

$$E_z = \frac{A_z}{f\, d^2}$$

U is typical velocities, d is typical depths,

Vertical mixing is considerably less than horizontal mixing because the ocean is stratified

As depth increases, the frictional force becomes much smaller than Coriolis force

Ekman v. Reality

- Inertial currents dominate
- Flow is nearly independent of depth within the mixed layer on time periods on the order of the inertial period (i.e. the mixed layer moves like a slab)
- Current shear is strongest at the top of the thermocline
- Flow averaged over many inertial periods is almost exactly that calculated by Ekman
- Ekman depth is typically on target with experiments, but velocities are often as much as half the calculated value
- Angle between wind and flow at surface depends on latitude and is near 45 degrees at mid-latitudes

Ekman Mass Transport

- Integral of the Ekman Velocities down to a depth d:

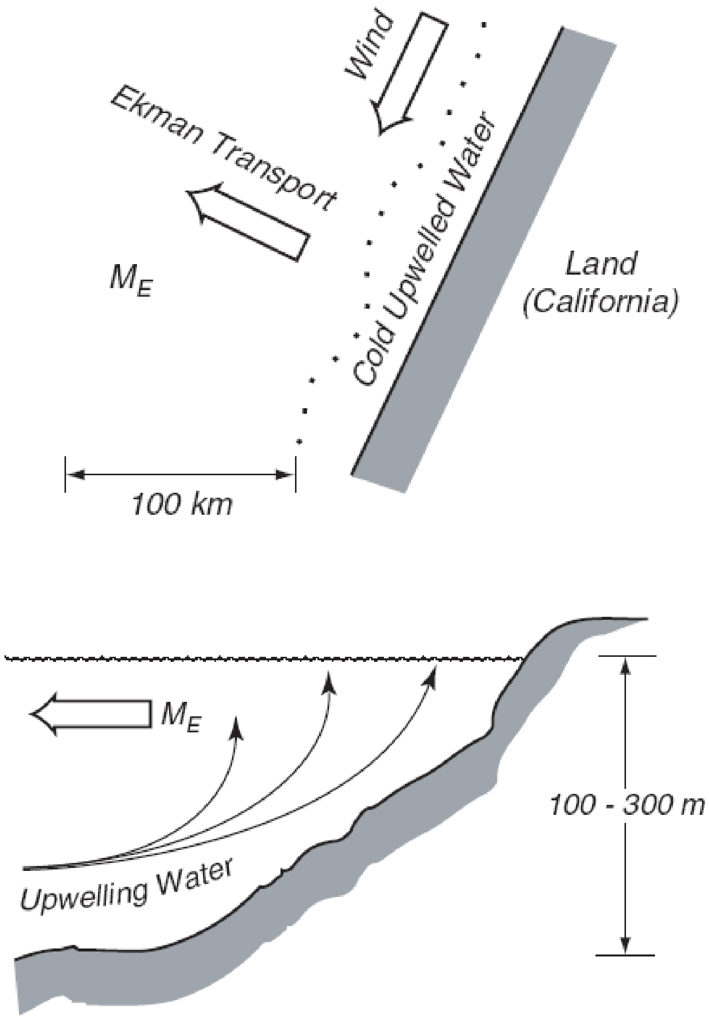
$$M_{Ex} = \int_{-d}^{0} \rho U_E \, dz, \qquad M_{Ey} = \int_{-d}^{0} \rho V_E \, dz$$

- Ekman transport relates the surface wind stress:

$$f\, M_{Ey} = -T_{xz}(0)$$
$$f\, M_{Ex} = \quad T_{yz}(0)$$

- Mass transport is perpendicular to wind stress
- In the northern hemisphere, *f* is positive, and the mass transport is in the *x* direction, to the east.

Coastal Upwelling



Courtesy of Prof. Robert Stewart. Used with permission.

- Upwelling enhances biological productivity, which feeds fisheries.
- Cold upwelled water alters local weather. Weather onshore of regions of upwelling tend to have fog, low stratus clouds, a stable stratified atmosphere, little convection, and little rain.
- Spatial variability of transports in the open ocean leads to upwelling and downwelling, which leads to redistribution of mass in the ocean, which leads to wind-driven geostrophic currents via *Ekman pumping*.

Source: Introduction to Physical Oceanography, http://oceanworld.tamu.edu/home/course_book.htm

Ekman Pumping

- The horizontal variability of the wind blowing on the sea surface leads to horizontal variability of the Ekman transports.
- Because mass must be conserved, the spatial variability of the transports must lead to vertical velocities at the top of the Ekman layer.
- To calculate this velocity, we first integrate the continuity equation in the vertical direction:

$$\rho \int_{-d}^{0} \left(\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} \right) dz = 0$$

$$\frac{\partial}{\partial x} \int_{-d}^{0} \rho\, u\, dz + \frac{\partial}{\partial y} \int_{-d}^{0} \rho\, v\, dz = -\rho \int_{-d}^{0} \frac{\partial w}{\partial z}\, dz$$

$$\frac{\partial M_{Ex}}{\partial x} + \frac{\partial M_{Ey}}{\partial y} = -\rho \left[w(0) - w(-d) \right]$$

- By definition, the Ekman velocities approach zero at the base of the Ekman layer, and the vertical velocity at the base of the layer $w_E(-d)$, due to divergence of the Ekman flow, must be zero.

Vertical Ekman Velocity

$$\frac{\partial M_{Ex}}{\partial x} + \frac{\partial M_{Ey}}{\partial y} = -\rho\, w_E(0)$$

$$\nabla_H \cdot \mathbf{M}_E = -\rho\, w_E(0)$$

- Where M_E is the vector mass transport due to Ekman flow in the upper boundary layer of the ocean, and ∇_H is the horizontal divergence operator.

- This states that the horizontal divergence of the Ekman transports leads to a vertical velocity in the upper boundary layer of the ocean, a process called *Ekman Pumping*

Ekman Pumping and Wind Stress

- If we use the Ekman mass transports in the previous equations we can relate Ekman pumping to the wind stress, **T**.

$$w_E(0) = -\frac{1}{\rho}\left[\frac{\partial}{\partial x}\left(\frac{T_{yz}(0)}{f}\right) - \frac{\partial}{\partial y}\left(\frac{T_{xz}(0)}{f}\right)\right]$$

$$w_E(0) = -\mathrm{curl}\left(\frac{\mathbf{T}}{\rho f}\right)$$

- The vertical velocity at the sea surface *w(0)* must be zero because the surface cannot rise into the air, so $w_E(0)$ must be balanced by another vertical velocity.
- This is balanced by a geostrophic velocity $w_G(0)$ at the top of the interior flow in the ocean. (Next lecture!)