MASSACHUSETTS INSTITUTE OF TECHNOLOGY
Department of Mechanical Engineering

2.004 Dynamics and Control II
Fall 2007

Problem Set #3

Solution Posted: Friday, Sept. 28, '07

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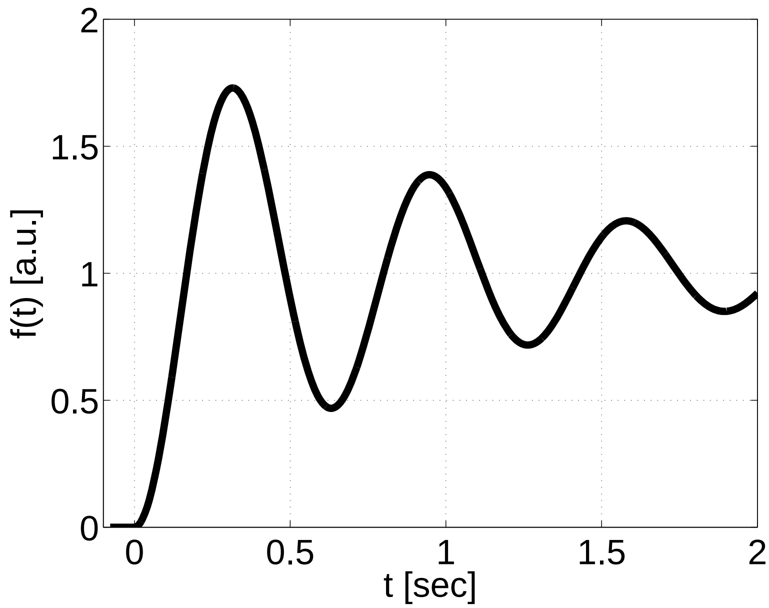
1. A second–order system has the step response shown below.[1] Determine its transfer function.



Answer: This is under–damped 2nd order system. Starting from the transfer function of the second order system

$$A\frac{\omega_n^2}{s^2+2\zeta\omega_n s+\omega^2},$$

we have to decide the parameters of A(constant),ζ(damping ratio) and ω_n(natural frequency).

From the final value theorem,

$$\lim_{s\to 0}\frac{1}{s}s\frac{A\omega_n^2}{s^2+2\zeta\omega_n s+\omega_n^2}=A$$

[1]a.u. denotes arbitrary units; its use appropriate when we consider a function that does not correspond to any particular physical quantity.

and the steady state value is 1 (from the given figure). Therefore, $A = 1$.

The step response of the under–damped second order system is

$$\left[1 - ae^{-\sigma_d t}\cos\left(\omega_d t - \phi\right)\right] u(t),$$

where $\sigma_d = \zeta\omega_n$ and $\omega_d = \omega_n\sqrt{1-\zeta^2}$.

From the lecture note 7 (pp. 26), %OS $= \exp\left(-\frac{\zeta\pi}{1-\zeta^2}\right)$: 72%.

Thus the damping ratio $\zeta \approx 0.1$.

To get the natural frequency, we choose two peak points at $t_1 = 0.35$ sec and $t_2 = 0.95$ sec. The cosine term will be 1 at the peaks, so that we can consider exponential decay term only.

$$\begin{aligned} f(t_1) &= 1 - ae^{-\sigma_d t_1} = 1.72 \\ f(t_2) &= 1 - ae^{-\sigma_d t_2} = 1.4 \end{aligned}$$

Dividing the two equations, we obtain

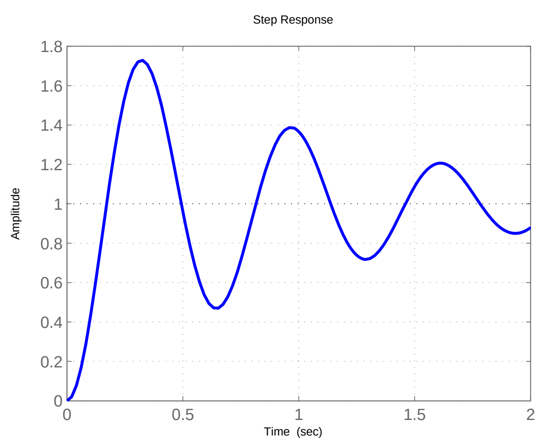
$$\frac{ae^{-\sigma_d t_1}}{ae^{-\sigma_d t_1}} = \frac{1 - 1.72}{1 - 1.4}.$$

From that $\sigma_d = \left\{\ln\left(\frac{0.72}{0.4}\right)\right\} / \left\{t_2 - t_1\right\} = 0.9796$. Therefore $\omega_n \approx 9.8$. (The reason why I picked two points instead of one point is to cancel the constant a).

The transfer function is

$$\frac{64}{s^2 + 1.96s + 96},$$

and its step response by MATLAB is



Note that the estimated parameters might be slightly different than the original because our reading of the plot can never be completely accurate.

2. Consider again the system of a DC motor with a parallel current source connected via a gear pair to an inertia that we saw in Problem 5 of PS02. Substituting numerical values $i_s = 1.0u(t)$ A, $R = 5\,\Omega$, $K_m = 0.5\,\mathrm{N\cdot m/A}$, $K_v = 0.5\,\mathrm{V\cdot sec}$, $J_m = 0.1\,\mathrm{kg\cdot m^2}$, $(N_2/N_1) = 10$, $J = 6\,\mathrm{kg\cdot m^2}$, $K = 1\,\mathrm{N\cdot m/rad}$, derive and plot the step response for the following two cases:

a) $b = 9.4\,\mathrm{N\cdot m\cdot sec/rad}$;

b) $b = 0.76\,\mathrm{N\cdot m\cdot sec/rad}$.

Answer:

The transfer function is

$$\frac{\Theta(s)}{I_s(s)} = \frac{(N_2/N_1)K_m}{\left(J + \frac{N_2^2}{N_1^2}J_m\right)s^2 + \left(b + \frac{N_2^2}{N_1^2}\frac{K_vK_m}{R}\right)s + K}.$$

Rearranging it, we can re–write it as

$$\frac{\Theta(s)}{I_s(s)} = \frac{(N_2/N_1)K_m}{K}\frac{K/(J + (N_2/N_1)^2J_m)}{s^2 + \frac{bR + (N_2^2/N_1^2)K_vK_m}{R(J + (N_2^2/N_1^2)J_m)}s + \frac{K}{J + (N_2^2/N_1^2)J_m}}.$$

The general form of the transfer function is

$$\frac{\Theta(s)}{I_s(s)} = A\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where the natural frequency $\omega_n = \sqrt{\frac{K}{J + (N_2^2/N_1^2)J_m}}$ and the damping ratio $\zeta = \frac{bR + (N_2^2/N_1^2)K_vK_m}{R(J + (N_2^2/N_1^2)J_m)}\frac{1}{2\omega_n}$.

a) $b = 9.4\,\mathrm{N\cdot m\cdot sec/rad}$;

$\omega_n = 0.25$ and $\zeta = 1.8 > 1$. (Over–damped system) Transfer function is

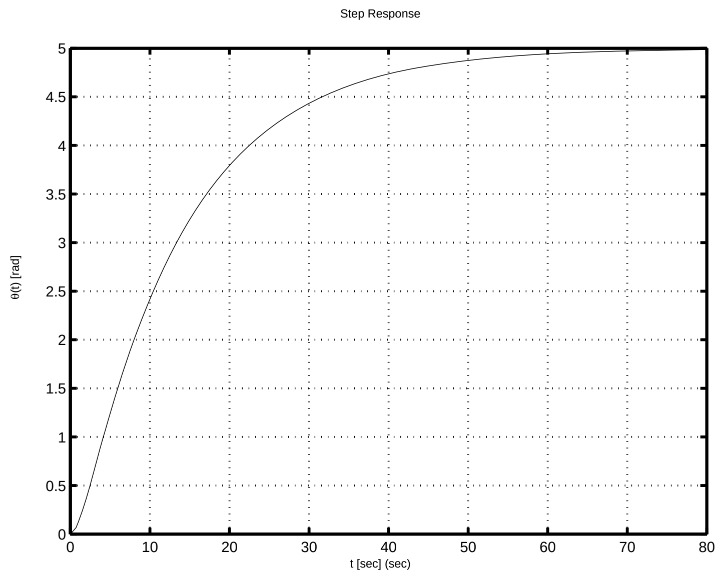
$$\frac{\Theta(s)}{I_s(s)} = \frac{5/16}{s^2 + (14.4/16)s + (1/4)^2}$$

whose poles are $p_1 = -0.8242$ and $p_2 = -0.0758$. To obtain its step response, we do partial fraction expansion.

$$A\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s} + \frac{K_2}{s + 0.8242} + \frac{K_3}{s + 0.0758},$$

where $K_1 = 5, K_2 = 0.5067, K_3 = -5.5067$. The step response is

$$f(t) = (K_1 + K_2e^{-0.8242t} + K_3e^{-0.0758t})u(t).$$



b) $b = 0.76\,\mathrm{N \cdot m \cdot sec/rad}$.

$\omega_n = 0.25$ and $\zeta = 0.72 < 1$. (Under–damped system) The transfer function is

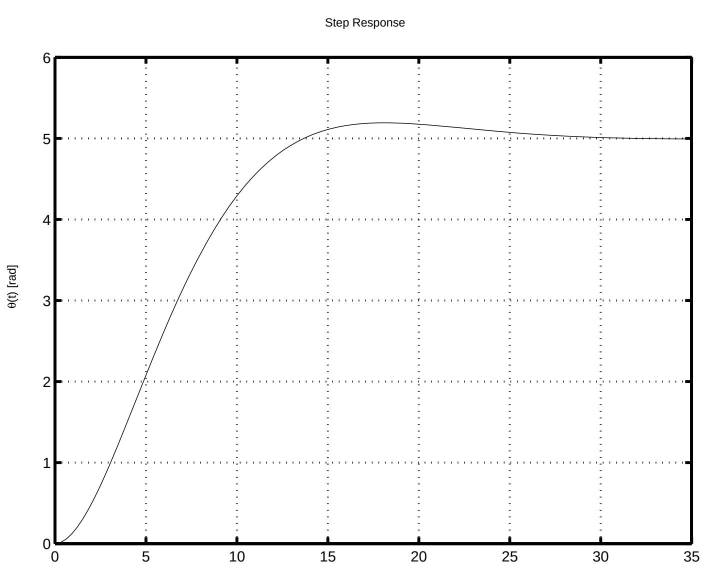
$$\frac{\Theta(s)}{I_s(s)} = \frac{5/16}{s^2 + (5.76/16)s + (1/4)^2}$$

whose poles are $p_1 = -0.18 + j0.1735$ and $p_1 = -0.18 - j0.1735$. Doing partial fraction expansion, we obtain

$$A\frac{\omega_n^2}{s^2 + 2\zeta\omega_n s + \omega_n^2} = \frac{K_1}{s} + \frac{K_2 s + K_3}{s^2 + 2\zeta\omega_n s + \omega_n^2},$$

where $K_1 = 5, K_2 = -5, K_3 = -1.8$. The step response is

$$f(t) = (K_1 + K_2 e^{-\sigma_d t}\cos(\omega_d t) + K_3 e^{-\sigma_d t}\sin(\omega_d t))u(t).$$

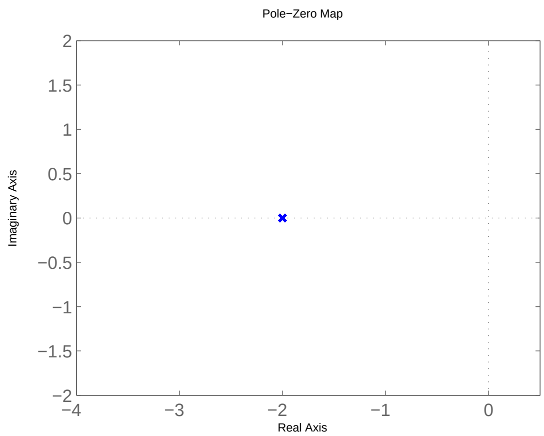


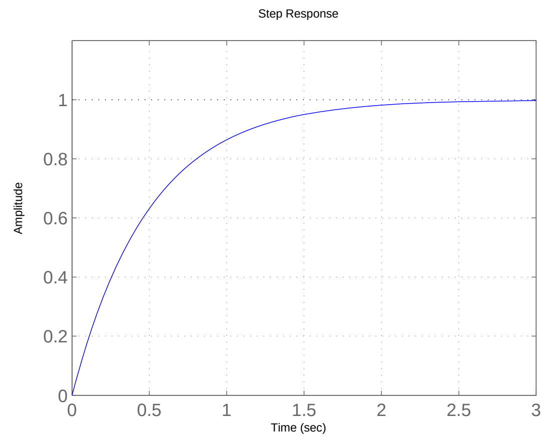
3. Problem 8 from Nise textbook, Chapter 4 (page 234).

Answer: ***Plotting the step response was not required; we did it here for completeness.***

a) $T(s) = \frac{2}{s+2}$

Answer: Pole: p=-2, Zero: none

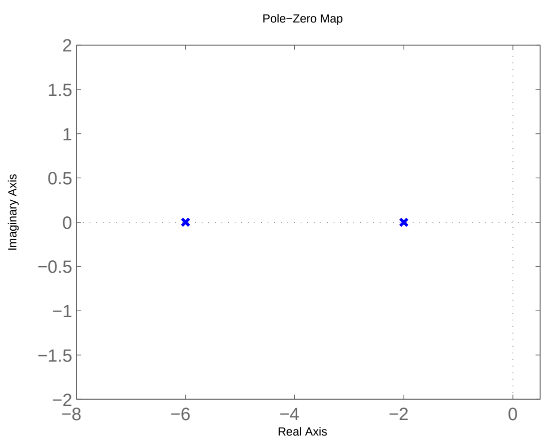


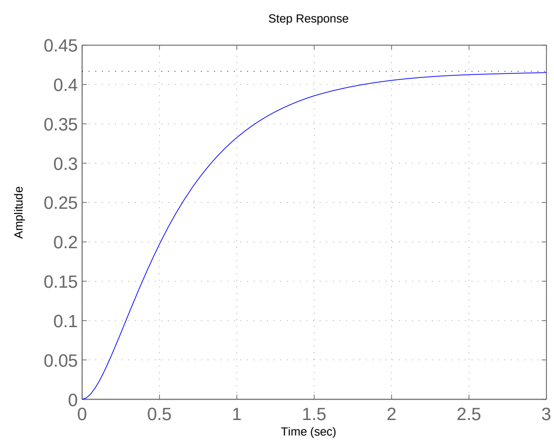


Step response $(1 - e^{-2t})u(t)$. 1st order system.

b) $T(s) = \frac{5}{(s+2)(s+6)}$

Answer: Poles: $p_1 = -2$, $p_2 = -6$, Zero: none



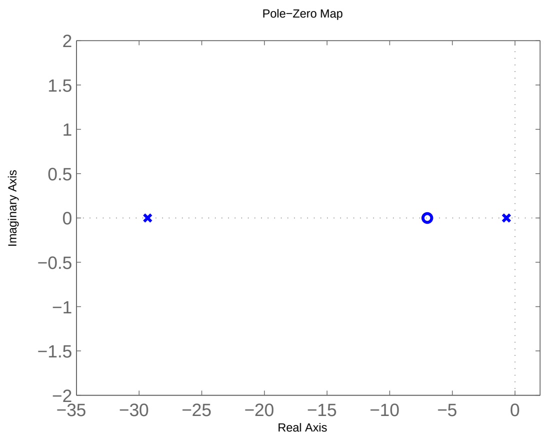


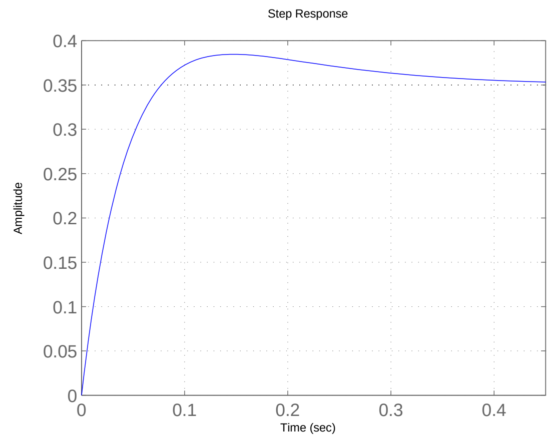
$\omega_n^2 = 12$ and $\zeta = 4/\sqrt{12} = 1.15 > 1$. 2nd order overdamped system.

Step response $\left[1 + K_1 e^{-p_1 t} + K_2 e^{-p_2 t}\right] u(t)$.

c) $T(s) = \frac{10(s+7)}{(s+10)(s+20)}$

Answer: Poles: $p_1 = -10$, $p_2 = -20$, Zeros: $z_1 = -7$

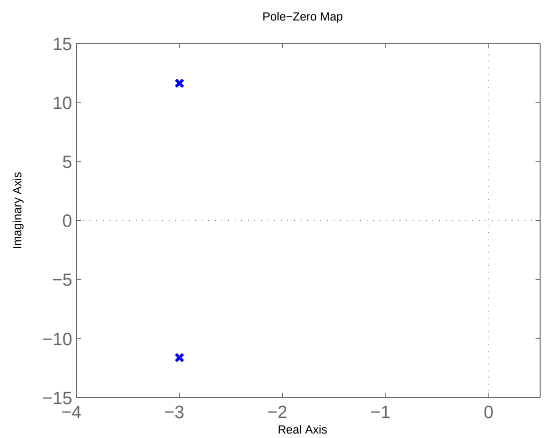


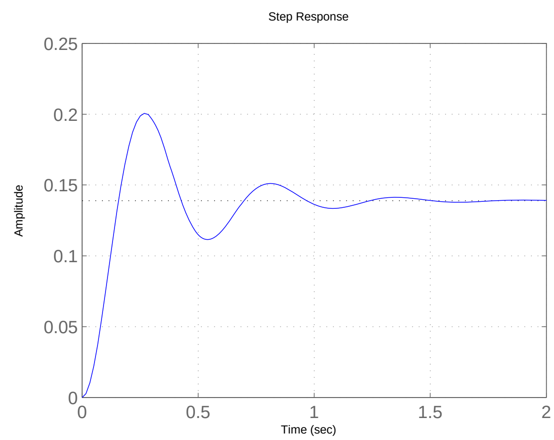


$\omega_n^2 = 200$ and $\zeta = 30/2/\sqrt{200} = 1.06 > 1$. 2nd order overdamped system. Step response $\left[1 + K_1 e^{-p_1 t} + K_2 e^{-p_2 t}\right] u(t)$.

d) $T(s) = \frac{20}{s^2+6s+144}$

Answer: Poles: $p_1 = -3 + j11.619$, $p_2 = -3 - j11.619$, Zeros: none

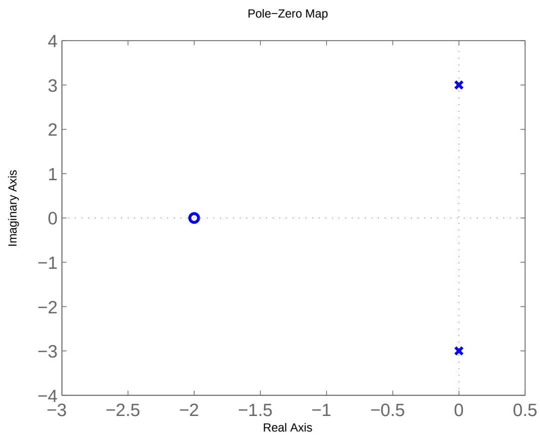


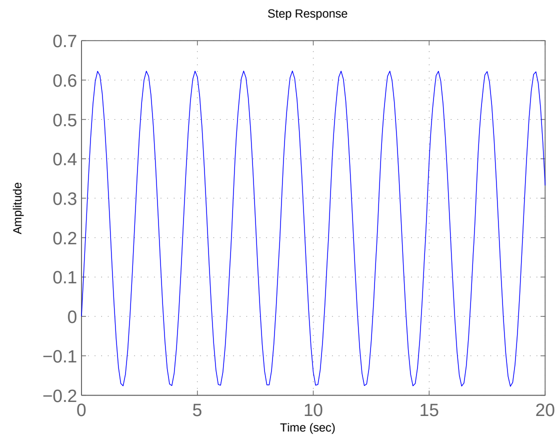


$\omega_n^2 = 144$ and $\zeta = 6/2/\sqrt{144} = 0.25 < 1$. 2nd order underdamped system. Step response $\left[1 - Ae^{-\sigma_d t} \cos\left(\omega_d t - \phi\right)\right] u(t)$.

e) $T(s) = \frac{s+2}{s^2+9}$

Answer: Poles: $p_1 = 3j$, $p_2 = -3j$, Zero: $z = -2$



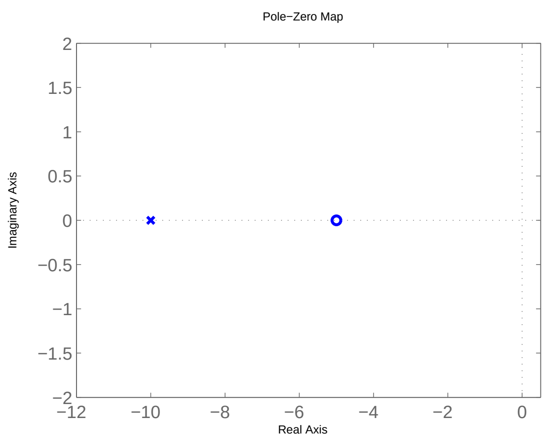


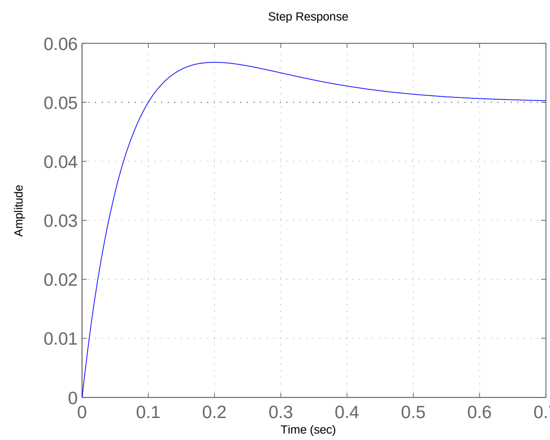
$\omega_n^2 = 9$ and $\zeta = 0$. 2nd order undamped system.

Step response $[1 - K_1 \sin(3t) + K_2 \cos(3t)]\, u(t)$.

f) $T(s) = \frac{(s+5)}{(s+10)^2}$

Answer: Poles: $p = -10(double)$, Zeors: $z = -5$





$\omega_n^2 = 100$ and $\zeta = 1$. 2nd order critically damped system.

Step response $[K_0 + K_1 e^{-10t} - K_2 t e^{-10t}]\, u(t)$.