2.004 Dynamics and Control II
Spring 2008

MASSACHUSETTS INSTITUTE OF TECHNOLOGY
DEPARTMENT OF MECHANICAL ENGINEERING

2.004 *Dynamics and Control II*
Spring Term 2008

Lecture 15[1]

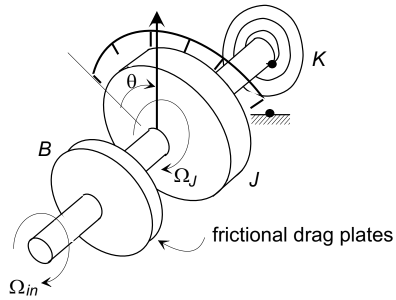
Reading:

- Class Handout: *Modeling Part 1: Energy and Power Flow in Linear Systems* Sec. 3.
- Class Handout: *Modeling Part 2: Summary of One-Port Primitive Elements*
- Nise: Secs. 2.4 and 2.6.

1 Rotational Systems (continued)

■ Example 1

The diagram shows a mechanical tachometer that uses frictional drag plates to create a torque proportional to angular velocity difference. The angular velocity is indicated by the displacement θ of a torsional spring.

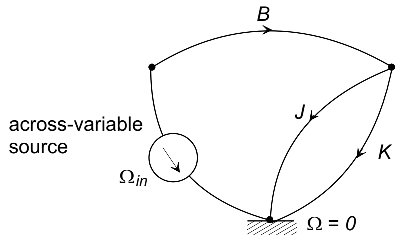


Find the transfer function relating the displacement of the indicator θ to the input angular velocity Ω_{in}

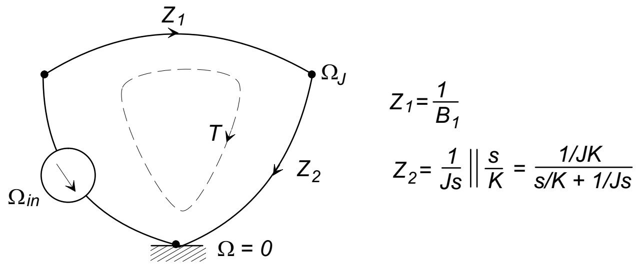
$$H(s) = \frac{\theta(s)}{\Omega_{in}(s)}$$

and show that for a constant input angular velocity the steady-state indicated speed $\theta_{ss} \propto \Omega_i$.

Solution: There are two distinct angular velocities, and the system graph is:



Using impedances, redraw the graph combining the inertia and the spring into a single impedance Z_2:



Then

$$\begin{aligned}\Omega_J &= \frac{Z_2}{Z_1 + Z_2}\Omega_{in}(s) \\ &= \frac{s/(Js^2+K)}{1/B + s/(Js^2+K)}\Omega_{in}(s) \\ &= \frac{Bs}{Js^2+Bs+K}\Omega_{in}(s).\end{aligned}$$

But the angular displacement $\theta(s) = \Omega_J(s)/s$ so that

$$\boxed{H(s) = \frac{B}{Js^2+Bs+K}}$$

If the input velocity is a step function at $t = 0$, the $\Omega_{in}(s) = \Omega/s$ and the steady-state indicated value will be

$$\theta_{ss} = \lim_{t\to\infty}\theta(t) = \lim_{s\to 0} s\theta(s) = \lim_{s\to 0} sH(s)\frac{\Omega}{s} = \frac{B}{K}\Omega$$

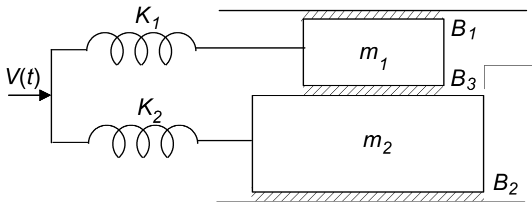
show that the indicated speed (θ) is proportional to the input angular velocity.

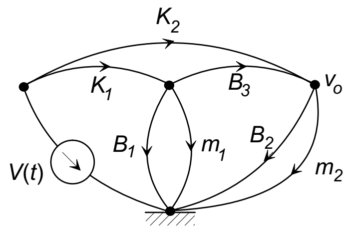
(This example is repeated using mesh equations below.)

2 Transfer Function Generation Using Mesh/Loop Equations

This method is useful when the system contains an across variable source.

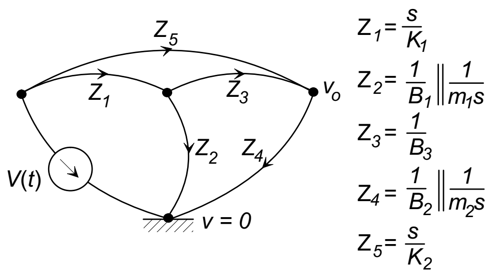
We will develop the method using the following mechanical system as an example.



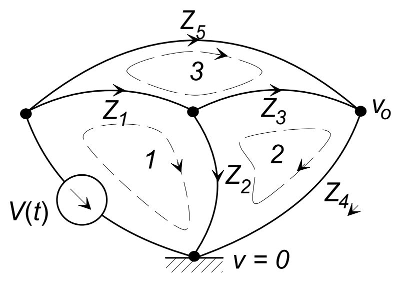


The input is a velocity source $V(t)$, and the output is the velocity of the mass m_2.

Step 1: (This step is optional.) Do some simple impedance combinations to reduce the complexity, being careful not to eliminate any nodes or branches that specify the output variable.



Step 2: Define a set of closed loops/meshes, making sure that each branch is covered by at least one loop.



Step 3: Write a loop equation for each loop:

$$
\begin{aligned}
v_{Z_1} + v_{Z_2} - V &= 0 \\
v_{Z_3} + v_{Z_4} - v_{Z_2} &= 0 \\
v_{Z_5} - v_{Z_3} - v_{Z_1} &= 0
\end{aligned}
$$

Step 4: At this point we assume that each loop has a continuous through variable with it, that is loop n has F_n associated with it. When a branch is contacted by more that one loop, *the branch through variable is the sum of all of the contacting loop through variables.* We use impedance relationships and substitute loop through variable expressions for the across variables:

$$\begin{aligned}
Z_1(F_1 - F_3) + Z_2(F_1 - F_2) - V &= 0 \\
Z_3(F_2 - F_3) + Z_4 F_2 - Z_2(F_1 - F_2) &= 0 \\
Z_5 F_3 - Z_3(F_2 - F_3) - Z_1(F_1 - F_3) &= 0
\end{aligned}$$

Step 5: Rewrite these equations collecting terms in the loop through variables to create a set of linear equations:

$$\begin{aligned}
(Z_1 + Z_2)F_1 - Z_2 F_2 - Z_1 F_3 &= V \\
-Z_2 F_1 + (Z_2 + Z_3 + Z_4)F_2 - Z_3 F_3 &= 0 \\
-Z_1 F_1 + Z_3 F_2 + (Z_1 + Z_3 + Z_5)F_3 &= 0
\end{aligned}$$

It may be useful to express these equations in matrix form:

$$\begin{bmatrix} (Z_1 + Z_2) & -Z_2 & -Z_1 \\ -Z_2 & (Z_2 + Z_3 + Z_4) & -Z_3 \\ -Z_1 & Z_3 & (Z_1 + Z_3 + Z_5) \end{bmatrix} \begin{bmatrix} F_1 \\ F_2 \\ F_3 \end{bmatrix} = \begin{bmatrix} V \\ 0 \\ 0 \end{bmatrix}$$

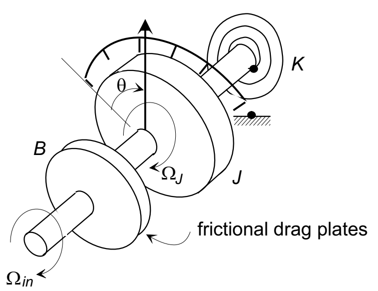
Step 6: Identify how the output variable is related to the loop through variables. In this case

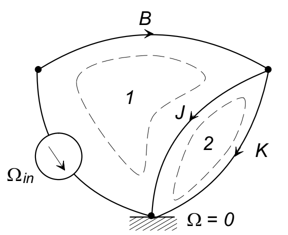
$$v_{out} = F_2 Z_4$$

Step 7: Solve the set of linear equations for the variable(s) identified in Step 6, in this case F_2, using any method, and substitute in the output equation (Step 6).
(we will not do this step here).

■ Example 2

Repeat the mechanical tachometer problem of Example 1, this time using mesh equations.





With the system graph and two loops defined as above, the loop equations are:

$$\begin{aligned} \Omega_B + \Omega_J - \Omega_{in} &= 0 \\ \Omega_K - \Omega_J &= 0 \end{aligned}$$

Substitute for the across variables (Ω):

$$\begin{aligned} Z_B T_1 + Z_K(T_1 - T_2) &= \Omega_{in} \\ Z_J T_2 - Z_K(T_1 - T_2) &= 0 \end{aligned}$$

Rearrange:

$$\begin{aligned} (Z_B + Z_K)T_1 - Z_K T_2 &= \Omega_{in} \\ -Z_K(T_1 + (Z_J + Z_K)T_2 &= 0 \end{aligned}$$

and in matrix form

$$\begin{bmatrix} Z_B + Z_K & -Z_K \\ -Z_K & Z_J + Z_K \end{bmatrix} \begin{bmatrix} T_1 \\ T_2 \end{bmatrix} = \begin{bmatrix} \Omega_{in} \\ 0 \end{bmatrix}$$

The output variable is $\Omega_J = Z_J T_2$, therefore use Cramer's rule to solve for T_2.

$$T_2 = \frac{\begin{vmatrix} Z_B + Z_K & \Omega_{in} \\ -Z_K & 0 \end{vmatrix}}{\begin{vmatrix} Z_B + Z_K & -Z_K \\ -Z_K & Z_J + Z_K \end{vmatrix}} = \frac{Z_K \Omega_{in}}{Z_B Z_J + Z_B Z_K + Z_K Z_J}$$

and since $\Omega_J = Z_J T_2$, $Z_J = 1/(Js)$, $Z_K = s/K$, and $Z_B = 1/B$

$$\Omega_J = \frac{Z_J Z_K \Omega_{in}}{Z_B Z_J + Z_B Z_K + Z_K Z_J} = \frac{Bs}{Js^2 + Bs + K}\Omega_{in}$$

and as before, $\theta(s) = \Omega_J / s$ giving

$$\boxed{H(s) = \frac{\theta(s)}{\Omega_{in}(s)} = \frac{B}{Js^2 + Bs + K}}$$

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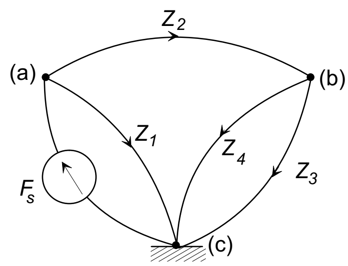
3 Transfer Function Generation Using Node Equations

This method is useful when the system contains a through variable source.

We will demonstrate the method using the following graph:



Step 1: Write node equations for each node (except the reference node) - let f be a generalized through variable.

$$
\begin{aligned}
F_s - f_{Z_1} - f_{Z_2} &= 0 \qquad \text{node(a)} \\
f_{Z_2} - f_{Z_4} - f_{Z_3} &= 0 \qquad \text{node(b)}
\end{aligned}
$$

Step 2: Express the branch through variables in terms of admittances ($Y = 1/Z$) and across variables

$$
\begin{aligned}
(v_a - v_b)Y_2 + (v_a - v_c)Y_1 &= F_s \\
(v_a - v_b)Y_2 + (v_b - v_c)Y_4 - (v_b - v_c)Y_3 &= 0
\end{aligned}
$$

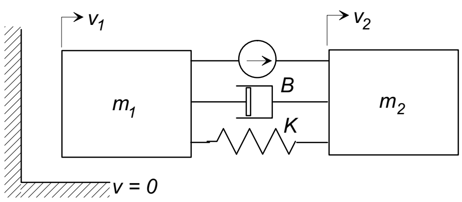
Step 3: Collect terms in the across variables, and form a set of linear equations (note that node (c) is the reference node, therefore $v_c = 0$).

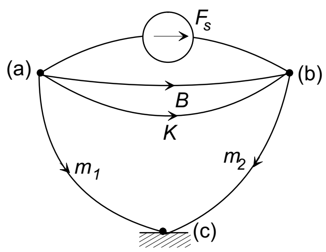
$$
\begin{aligned}
(Y_1 + Y_2)v_a - Y_2 v_b &= F_s \\
Y_2 v_a - (Y_2 + Y_3 + Y_4)v_b &= 0
\end{aligned}
$$

Step 4: Identify the output variable in terms of nodal across variables and solve the linear equations.

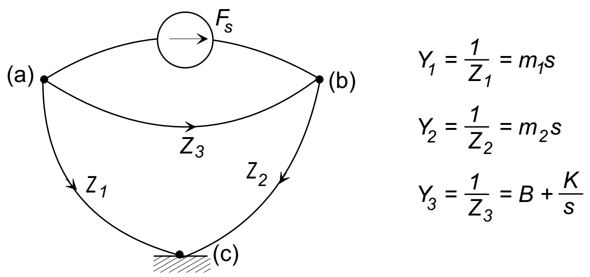
■ Example 3

Find $H(s) = X_{m_2}(s)/F(s)$ for the following system:





The graph has been simplified by combining parallel elements:



At the nodes (a) and (b):

$$\begin{aligned} -F_B - F_{m_1} - F_s &= 0 \\ F_B - F_{m_2} + F_s &= 0 \end{aligned}$$

Substitute for through variables

$$\begin{aligned} -(v_a - v_b)Y_3 - v_a Y_1 &= F_s \\ (v_a - v_b)Y_3 - v_b Y_1 &= -F_s \end{aligned}$$

Reorganize and write in matrix form:

$$\begin{bmatrix} -(Y_3 + Y_1) & Y_3 \\ Y3 & -(Y_2 + Y_3) \end{bmatrix} \begin{bmatrix} v_a \\ v_b \end{bmatrix} = \begin{bmatrix} F_s \\ -F(s) \end{bmatrix}$$

Recognize that the output $x_{m_2}(s) = v_b(s)/s$, use Cramer's rule to solve for v_b:

$$v_b = \frac{\begin{vmatrix} -(Y_3 + Y_1) & F(s) \\ Y3 & -F_s) \end{vmatrix}}{\begin{vmatrix} -(Y_3 + Y_1) & Y_3 \\ Y3 & -(Y_2 + Y_3) \end{vmatrix}} = \frac{Y_1 F_s}{Y_1 Y_2 + Y_3(Y_1 + Y_2)}$$

$Y_1 = m_1 s$, $Y_2 = m_2 s$, and $Y_3 = B + K/s$, giving

$$v_{m_2}(s) = v_b(s) = \frac{m_1 s}{m_1 m_2 s^2 + B(m_1 + m_2)s + K(m_1 + m_2)} F_s(s)$$

and

$$\boxed{H(s) = \frac{x_{m_2}(s)}{F_s(s)} = \frac{1}{s}\frac{v_{m_2}(s)}{F_s(s)} = \frac{m_1}{m_1 m_2 s^2 + B(m_1 + m_2)s + K(m_1 + m_2)}}$$
