2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Thomas Peacock
5/2/2007

Lecture 20

Vibrations: Second Order Systems with One Degree of Freedom, Free Response

Single Degree of Freedom System

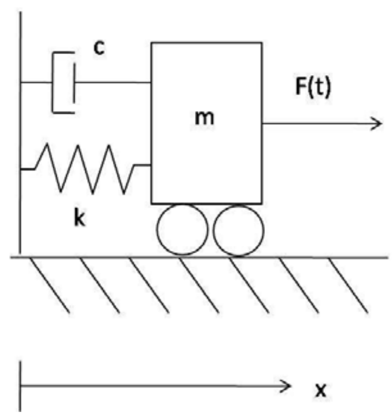

Figure 1: Cart attached to spring and dashpot. Figure by MIT OCW.

$$\boxed{m\ddot{x} + c\dot{x} + kx = F(t)}$$

System response? What is $x(t)$?

Use 18.03 Background.

$$x(t) = \underbrace{\text{Free Response}}_{\text{Complementary Solution, when } F(t)=0} + \underbrace{\text{Response Due to Forcing}}_{\text{Particular Solution}}$$

This lecture will cover the Free Response.

Free Response

Look at $k \to 0$

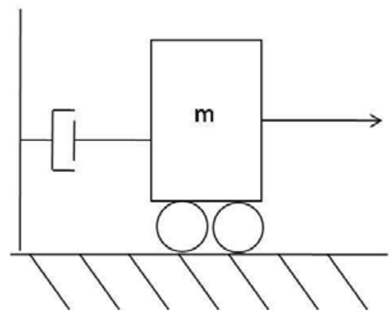

Figure 2: Cart with dashpot only. Figure by MIT OCW.

$$m\ddot{x} + c\dot{x} = 0$$

Assume conditions $x(0) = x_0$ and $\dot{x}(0) = v_0$.

$$m\ddot{x} + c\dot{x} = m\dot{v} + cv = 0$$

$$v = v_0 e^{(-ct/m)} \text{ already used } \dot{x}(0) = v_0$$

Integrate $v(t)$ once. Using $x(0) = x_0$, we obtain:

$$x = x_0 + \frac{mv_0}{c}\left(1 - e^{-\frac{c}{m}t}\right)$$

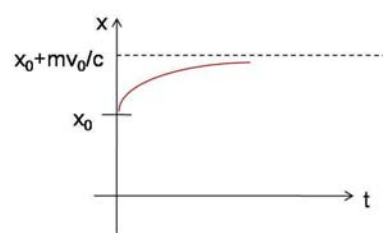

Figure 3: Solution to differential equation. Solution attenuates to a steady state value. Figure by MIT OCW.

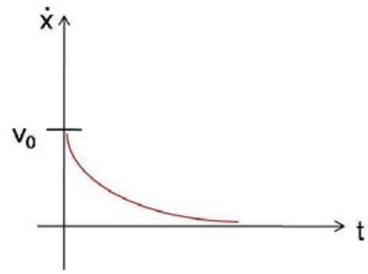

Figure 4: Velocity profile of solution. Velocity attenuates to zero. Figure by MIT OCW.

No oscillations. Because $k = 0$, there was no restoring term.

Look at $m \to 0$

$$c\dot{x} + kx = 0$$

or

$$\dot{x} = -\frac{k}{c}x$$

$$x(0) = x_0$$

Therefore:

$$x(t) = x_0 e^{-\frac{k}{c}t}$$

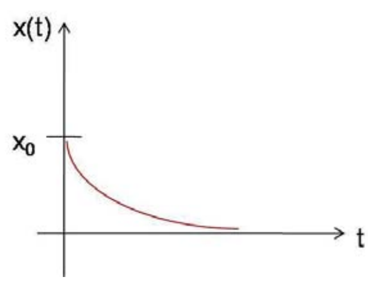

Figure 5: Solution to differential equation. Position decays to zero. Figure by MIT OCW.

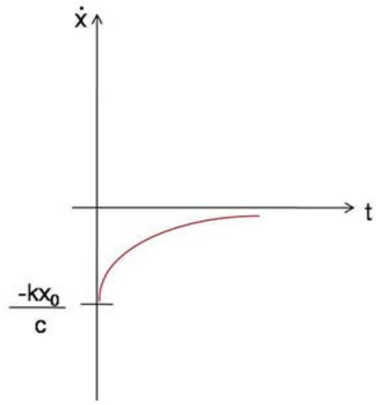

Figure 6: Velocity profile of solution. Value attenuates to steady state value. Figure by MIT OCW.

$$\dot{x} = -\frac{kx_0}{c} e^{-\frac{k}{c}t}$$

No oscillations in this system.
Dashpot force balances the spring force as $x \to 0$, spring force $\to 0$.

Vibrations require a restoring force (e.g. spring) and inertia (e.g. mass).

Full Free Response Problem

So let us consider the full problem:

$$\boxed{m\ddot{x} + c\dot{x} + kx = 0} \tag{1}$$

Note that $c\dot{x}(c > 0)$ is a damping term and is responsible for decay of oscillations.

Examination of Energy

$$\frac{d}{dt}(T+V) = \frac{d}{dt}\left(\frac{1}{2}m\dot{x}^2 + \frac{1}{2}kx^2\right) = m\dot{x}\ddot{x} + kx\dot{x} = \dot{x}(m\ddot{x} + kx) = \dot{x}(-c\dot{x}) = -c\dot{x}^2$$

For $c > 0$:

$$\frac{d}{dt}(T+V) < 0$$

Damping. Mechanical energy is dissipated.

For $c < 0$:

$$\frac{d}{dt}(T+V) > 0$$

Energy input (Control system providing energy)

Solution of the Equation with Engineering Quantities

Rewrite

$$m\ddot{x} + c\dot{x} + kx = 0$$

as:

$$\boxed{\ddot{x} + 2\zeta\omega_n\dot{x} + \omega_n^2 x = 0} \quad (2)$$

$$\omega_n^2 = \frac{k}{m}$$

$$\zeta = \frac{c}{2m\omega_n}$$

ω_n: Natural Frequency
ζ: Damping Ratio

To solve, we assume a solution of the form $x = Ae^{(\lambda t)}$

Substitute in Equation (2):

$$\lambda^2 + 2\zeta\omega_n\lambda + \omega_n^2 = 0$$

$$\boxed{\lambda = -\zeta\omega_n \pm \omega_n\sqrt{\zeta^2 - 1}} \quad (3)$$

When $\zeta^2 > 1$ and $\zeta^2 < 1$, the behavior is different.

Assume $c \geq 0$. ($\zeta \geq 0$) We have the following cases.

Case 1: Overdamped

$$\zeta > 1 \Rightarrow \lambda_1, \lambda_2 = \text{Real Negative Numbers}$$

$$x = A_{\pm}e^{\left(-\zeta\omega_n \pm \sqrt{\zeta^2-1}\right)} \rightarrow 0 \text{ as } t \rightarrow \infty$$

Case 2: Critically Damped

$$\zeta = 1 \Rightarrow \lambda_1, \lambda_2 = -\omega_n$$

$$x = (A_1 + A_2 t)e^{-\omega_n t} \rightarrow 0 \text{ as } t \rightarrow \infty \quad (4)$$

Equation (4) is the fastest approach to the set point. That is why it is named critically damped.

Case 3: Underdamped

$$0 \leq \zeta < 1$$

$$\lambda_1, \lambda_2 = -\zeta\omega_n \pm i\omega_d$$

$$\omega_d = \omega_n\sqrt{1-\zeta^2}$$

Underdamped (Not enough damping to prevent oscillations). When $\zeta \to 0$, $\omega_d \to \omega_n$ (Natural frequency).

$$x = \left[A_1 e^{i\omega_d t} + A_2 e^{-i\omega_d t}\right] e^{-\zeta\omega_n t}$$

Must have that A_1 and A_2 are complex conjugates because x is real.

$$\begin{aligned} x =& [A_1(\cos\omega_d t + i\sin\omega_d t) + A_2(\cos\omega_d t - i\sin\omega_d t)]e^{-\zeta\omega_n t} \\ =& [\underbrace{(A_1 + A_2)}_{A_3}\cos\omega_d t + \underbrace{i(A_1 - A_2)}_{A_4}\sin\omega_d t]e^{-\zeta\omega_n t} \end{aligned}$$

$$A_1 + A_2 = A_3$$

$$i(A_1 - A_2) = A_4$$

$$x = A_3\left[\cos\omega_d t + \frac{A_4}{A_3}\sin\omega_d t\right]e^{-\zeta\omega_n t}$$

$$x = A_3\left[\cos\omega_d t + \tan\phi\sin\omega_d t\right]e^{-\zeta\omega_n t}$$

$$x = \frac{A_3}{\cos\phi}\left[\cos\omega_d t\cos\phi + \sin\omega_d t\sin\phi\right]e^{-\zeta\omega_n t}$$

Note the trigonometric identity.

$$x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi) \tag{5}$$

$e^{-\zeta\omega_n t}$: Decaying in time
$\cos(\omega_d t - \phi)$: Oscillatory Behavior

C and ϕ can be found from initial conditions.

$$C = \frac{A_3}{\cos\phi} \tag{6}$$

$$\phi = \arctan\frac{A_4}{A_3} \tag{7}$$

Equations (6) and (7) relate C and ϕ to A_3 and A_4.

But $\frac{1}{\cos^2\phi} = 1 + \tan^2\phi$.

$$\frac{1}{\cos^2\phi} = 1 + \frac{A_4^2}{A_3^2}$$

$$\frac{1}{\cos\phi} = \frac{\sqrt{A_3^2 + A_4^2}}{A_3} \Rightarrow C = \sqrt{A_3^2 + A_4^2}$$

If $0 \leq \zeta < 1$, the solution will show decaying oscillations. How do we determine (C and ϕ) or (A_3 and A_4)? Often easier to relate A_3 and A_4 to initial conditions.

Initial Conditions: $x(0) = x_0$, $\dot{x}(0) = v_0$

$$x = [A_3 \cos\omega_d t + A_4 \sin\omega_d t]e^{-\zeta\omega_n t}$$

At $t = 0$, $x_0 = A_3$ (using $x(0) = x_0$)

$$\begin{aligned}\dot{x} =& [-A_3\omega_d \sin\omega_d t + A_4\omega_d \cos\omega_d t]e^{-\zeta\omega_n t} \\ &- \zeta\omega_n[A_3 \cos\omega_d t + A_4 \sin\omega_d t]e^{-\zeta\omega_n t}\end{aligned}$$

At $t = 0$:

$$v_0 = A_4\omega_d - \zeta\omega_n A_3 = A_4\omega_d - \zeta\omega_n x_0$$

$$A_4 = \frac{v_0 + \zeta\omega_n x_0}{\omega_d}$$

$$C = \sqrt{x_0^2 + \left(\frac{v_0 + \zeta\omega_n x_0}{\omega_d}\right)^2} \quad (8)$$

$$\tan\phi = \frac{v_0 + \zeta\omega_n x_0}{\omega_d x_0} \quad (9)$$

Examine solution.

$$x(t) = Ce^{-\zeta\omega_n t}\cos(\omega_d t - \phi)$$

$e^{-\zeta\omega_n t}$: Decay
$\cos(\omega_d t - \phi)$: Oscillating

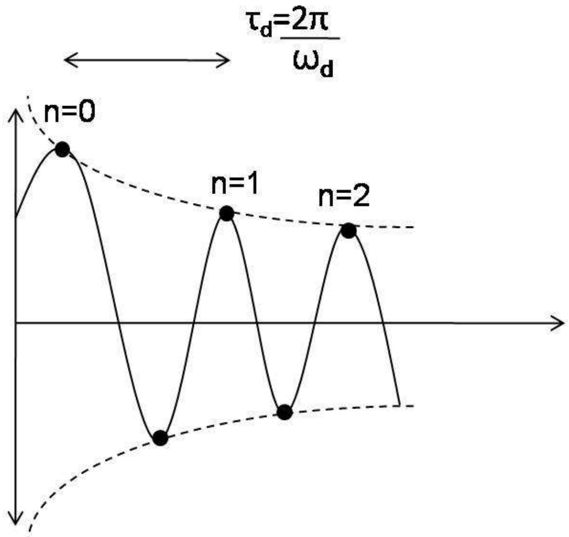

Figure 7: Solution both decays and oscillates given the presence of exponential solution and sinusoidal solution. Figure by MIT OCW.

Calculate Amplitude.

$$\frac{x(t)}{x(t+n\tau_d)} = \frac{e^{-\zeta\omega_n t}}{e^{[-\zeta\omega_n(t+n\tau_d)]}} = e^{\zeta\omega_n n\tau_d}$$

$$\ln\left[\frac{x(t)}{x(t+n\tau_d)}\right] = n\zeta\omega_n\tau_d = n\zeta\frac{\omega_n 2\pi}{\omega_d} = n\zeta\frac{\omega_n 2\pi}{\omega_n\sqrt{1-\zeta^2}} = n\zeta\frac{2\pi}{\sqrt{1-\zeta^2}} \qquad (10)$$

For $\zeta << 1$:

$$\ln\left[\frac{x(t)}{x(t+n\tau_d)}\right] = 2\pi n\zeta \qquad (11)$$

Need ω_n, ζ to define system.

Example Experiment: Flexible Rod.

Figure 8: Flexible rod. Figure by MIT OCW.

Measure frequency of oscillation: ω_d.
Measure amplitude over several periods to obtain $\frac{x(t)}{x(t+n\tau_d)}$. This ratio is related to the damping ratio ζ by the equations (10) or (11) if $\zeta << 1$.
With ω_d and ζ, one can calculate the natural frequency ω_n.