2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Thomas Peacock
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Lecture 15

Lagrangian Dynamics: Derivations of Lagrange's Equations

Constraints and Degrees of Freedom

Constraints can be prescribed motion

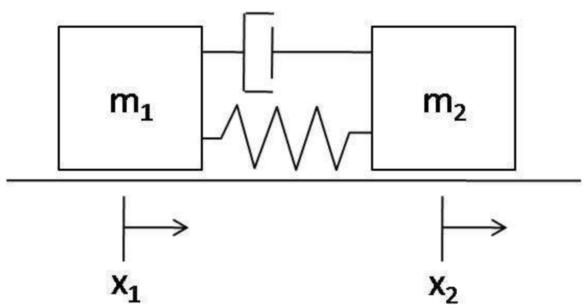

Figure 1: Two masses, m_1 and m_2 connected by a spring and dashpot in parallel. Figure by MIT OCW.

2 degrees of freedom

If we prescribe the motion of m_1, the system will have only 1 degree of freedom, only x_2. For example,

$$x_1(t) = x_0 \cos \omega t$$

$x_1 = x_1(t)$ is a constraint. The constraint implies that $\delta x_1 = 0$. The admissible variation is zero because position of x_1 is determined.
For this system, the equation of motion (use Linear Momentum Principle) is

$$m\ddot{x}_2 = -k(x_2 - x_1(t)) - c(\dot{x}_2 - \dot{x}_1(t))$$
$$m\ddot{x}_2 + c\dot{x}_2 + kx_2 = c\dot{x}_1(t) + kx_1(t)$$

$c\dot{x}_1(t) + kx_1(t)$: known forcing term
differential equation for $x_2(t)$: ODE, second order, inhomogeneous

Lagrange's Equations

For a system of n particles with ideal constraints

Linear Momentum

$$\dot{\underline{p}}_i = \underline{f}_i^{ext} + \underline{f}_i^{constraint} \tag{1}$$

$$\sum_{i=1}^{N} (\underline{f}_i^{ext} + \underline{f}_i^{constraint} - \dot{\underline{p}}_i) = 0 \tag{2}$$

$$\sum_{i=1} \underline{f}_i^{constraint} = 0$$

D'Alembert's Principle

$$\sum_{i=1}^{N} (\underline{f}_i^{ext} - \dot{\underline{p}}_i) \cdot \delta \underline{r}_i = 0 \tag{3}$$

Choose $\dot{p}_i = 0$ at equilibrium. We have the principle of virtual work.

Hamilton's Principle

Now $\dot{\underline{p}}_i = m_i \ddot{r}_i$, so we can write:

$$\sum_{i=1}^{N} (m_i \ddot{\underline{r}}_i - \underline{f}_i^{ext}) \cdot \delta \underline{r}_i = 0 \tag{4}$$

$$\delta W = \sum_{i=1}^{N} \underline{f}_i^{ext} \cdot \delta \underline{r}_i, \tag{5}$$

which is the virtual work of all active forces, conservative and nonconservative.

$$\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{i=1}^{N} m_i \left[\frac{d}{dt} (\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \dot{\underline{r}}_i \cdot \delta \dot{\underline{r}}_i \right] \tag{6}$$

(6) is obtained by using $\frac{d}{dt}(\dot{\underline{r}} \cdot \delta \underline{r}) = \ddot{\underline{r}} \delta \underline{r} + \dot{\underline{r}} \delta \dot{\underline{r}}$

$\dot{\underline{r}}_i \cdot \delta \dot{\underline{r}}_i$ can be rewritten as $\frac{1}{2}\delta(\dot{\underline{r}} \cdot \underline{r})$ by using $\delta(\dot{\underline{r}} \cdot \dot{\underline{r}}) = 2\dot{\underline{r}}\delta\dot{\underline{r}}$.

Substituting this in (6), we can write:

$$\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i = \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \delta \sum_{i=1}^{N} \frac{1}{2} m (\dot{\underline{r}}_i \cdot \dot{\underline{r}}_i) \tag{7}$$

The second term on the right is a kinetic energy term.

$$\delta \sum_{i=1}^{N} \frac{1}{2} m (\dot{\underline{r}}_i \cdot \dot{\underline{r}}_i) = \delta(\text{Kinetic Energy}) = \delta T$$

Now we rewrite (4) as:

$$\sum_{i=1}^{N} m_i \ddot{\underline{r}}_i \cdot \delta \underline{r}_i - \sum_{i=1}^{N} \underline{f}_i^{ext} \cdot \delta r_i = 0 \tag{8}$$

Substitute (5) and (7 into (8) to obtain:

$$\sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) - \delta T - \delta W = 0$$

Rearrange to give

$$\delta T + \delta W = \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i) \tag{9}$$

Integrate (9) between two definite states in time $\underline{r}(t_1)$ and $\underline{r}(t_2)$

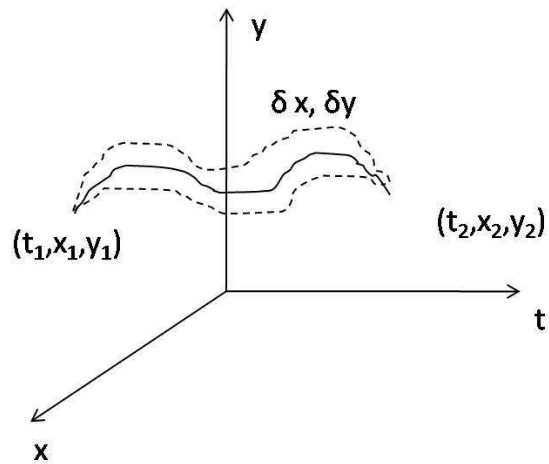

Figure 2: Between t_1 and t_2, there are admissible variations δx and δy. We are integrating over theoretically admissible states between t_1 and t_2 that satisfy all constraints. Figure by MIT OCW.

$$\int_{t_1}^{t_2} (\delta W + \delta T)dt = \int_{t_1}^{t_2} \sum_{i=1}^{N} m_i \frac{d}{dt}(\dot{\underline{r}}_i \cdot \delta \underline{r}_i)dt \tag{10}$$

$$= \sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i \Big|_{t_1}^{t_2} \tag{11}$$

The right hand side, $\sum_{i=1}^{N} m_i \dot{\underline{r}}_i \cdot \delta \underline{r}_i \Big|_{t_1}^{t_2} = 0$.

Why? $\dot{\underline{r}}_i \cdot \delta \underline{r}_i \Big|_{t_1}^{t_2} = 0$, because at a particular time, $\delta \underline{r}_i(t_i) = 0$. Also, we know the initial and final states. It is the behavior in between that we want to know.

The result is the *extended Hamilton Principle.*

$$\boxed{\int_{t_1}^{t_2} (\delta W + \delta T)dt = 0} \tag{12}$$

Generalized Fores and the Lagrangian

$$\delta W = \delta W^{conservative} + \delta W^{nonconservative} = -\delta V + \sum_{j=1}^{m} Q_j \delta q_j$$

Conservative δW:

$$\delta W = \underline{f}_i^{cons} \cdot \delta \underline{r}_i$$

$$\underline{f}_i^{cons} = -\frac{\partial V}{\partial \underline{r}_i}$$

$$\delta W = -\frac{\partial V}{\partial \underline{r}_i} \cdot \delta \underline{r}_i = -\delta V$$

Nonconservative δW:

$$Q_j \delta q_j$$

$$\sum_{j=1}^{m} Q_j \delta q_j$$

m: Total number of generalized coordinates
$Q_j = \Xi_j$: Generalized force for nonconservative work done
$q_j = \xi_j$: Generalized coordinate

Substitute for δW in (12) to obtain:

$$\int_{t_1}^{t_2} (\delta T - \delta V + \sum_{j=1}^{m} Q_j \delta q_j) dT = 0 \tag{13}$$

Define *Lagrangian*

$$\boxed{L = T - V}$$

The Lagrangian is a function of all the generalized coordinates, the generalized velocities, and time:

$$L = L(q_j, \dot{q}_j, t) \text{ where } j = 1, 2, 3 \ldots, m$$

(13) can now be written as

$$\int_{t_1}^{t_2} \left[\delta L(q_j, \dot{q}_j, t) + \sum_{j=1}^{m} Q_j \delta q_j \right] dt = 0 \tag{14}$$

Lagrange's Equations

We would like to express $\delta L(q_j, \dot{q}_j, t)$ as (a function) $\cdot\, \delta q_j$, so we take the total derivative of L. Note δt is 0, because admissible variation in space occurs at a fixed time.

$$\delta L = \sum_{j=1}^{m} \left[\left(\frac{\partial L}{\partial q_j}\right)\delta q_j + \left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta \dot{q}_j + \left(\frac{\partial L}{\partial t}\right)\delta t \right]$$

$$\int_{t_1}^{t_2} (\delta L) dt = \int_{t_1}^{t_2} \sum_{j=1}^{m} \left[\left(\frac{\partial L}{\partial q_j}\right)\delta q_j + \left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta \dot{q}_j \right] dt \tag{15}$$

To remove the $\delta \dot{q}_j$ in (15), integrate the second term by parts with the following substitutions:

$$\begin{aligned} u &= \left(\frac{\partial L}{\partial \dot{q}_j}\right) \\ du &= \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \\ y &= \delta q_j \\ dy &= \delta \dot{q}_j \end{aligned}$$

$$\begin{aligned} \int_{t_1}^{t_2} \sum_{j=1}^{m} \left(\frac{\partial L}{\partial \dot{q}_j}\right) \delta \dot{q}_j dt &= \sum_{j=1}^{m} \int_{t_1}^{t_2} \left(\frac{\partial L}{\partial \dot{q}_j}\right) \delta \dot{q}_j dt \\ &= \sum_{j=1}^{m} \left\{ \left(\frac{\partial L}{\partial \dot{q}_j}\right) \delta q_j \Big|_{t_1}^{t_2} - \int_{t_1}^{t_2} \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) \delta q_j \right] dt \right\} \end{aligned}$$

$$\left(\frac{\partial L}{\partial \dot{q}_j}\right) \delta q_j \Big|_{t_1}^{t_2} = 0$$

$$\int_{t_1}^{t_2} \sum_{j=1}^{m} \left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta \dot{q}_j dt = -\int_{t_1}^{t_2} \sum_{j=1}^{m} \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j dt \tag{16}$$

Combine (14), (15), and (16) to get:

$$\int_{t_1}^{t_2} \sum_{j=1}^{m} \left[\left(\frac{\partial L}{\partial q_j}\right)\delta q_j - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)\delta q_j + Q_j \delta q_j \right] dt = 0$$

$$\int_{t_1}^{t_2} \sum_{j=1}^{m} \left[-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) + \left(\frac{\partial L}{\partial q_j}\right) + Q_j \right] \delta q_j dt = 0$$

dt has finite values.
δq_j are independent and arbitrarily variable in a holonomic system. They are finite quantities. Thus, for the integral to be equal to 0,

$$-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)+\left(\frac{\partial L}{\partial q_j}\right)+Q_j = 0$$

Equations of Motion (Lagrange):

$$\boxed{Q_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right)-\left(\frac{\partial L}{\partial q_j}\right)}$$

or:

$$\boxed{\Xi_j = \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{\xi}_j}\right)-\left(\frac{\partial L}{\partial \xi_j}\right)}$$

Where $Q_j = \Xi_j =$ generalized force, $q_j = \xi_j =$ generalized coordinate, $j =$ index for the m total generalized coordinates, and L is the Lagrangian of the system.

Although these equations were formally derived for a system of particles, the same is true for rigid bodies.

Example 1: 2-D Particle, Horizontal Plane

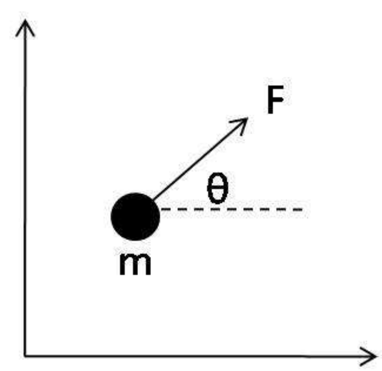

Figure 3: 2-D Particle on a horizontal plane subject to a force F. Figure by MIT OCW.

Cartesian Coordinates

$$q_1 = x$$
$$q_2 = y$$

$$\underline{r} = x\hat{\imath} + y\hat{\jmath}$$
$$\dot{\underline{r}} = \dot{x}\hat{\imath} + \dot{y}\hat{\jmath}$$

$$|\underline{v}|^2 = \dot{\underline{r}} \cdot \dot{\underline{r}} = \dot{x}^2 + \dot{y}^2 = \dot{q}_1^2 + \dot{q}_2^2$$

$$Q_1 = F\cos\theta$$
$$Q_2 = F\sin\theta$$

$$L = T - V$$

$$\begin{aligned} T &= \frac{1}{2}m(\dot{\underline{r}} \cdot \dot{\underline{r}}) \\ &= \frac{1}{2}m(\dot{q}_1^2 + \dot{q}_2^2) \end{aligned}$$

$V = 0$ (in horizontal plane, position with respect to gravity same at all locations)

For q_1 or (x)

$$\frac{\partial L}{\partial q_1} = 0$$
$$\frac{\partial L}{\partial \dot{q}_1} = m\dot{q}_1$$
$$\frac{d}{dt}\frac{\partial L}{\partial \dot{q}_1} = m\ddot{q}_1$$

$$\boxed{m\ddot{q}_1 - 0 = F\cos\theta}$$
$$\boxed{m\ddot{q}_2 = F\sin\theta}$$

Polar Coordinates

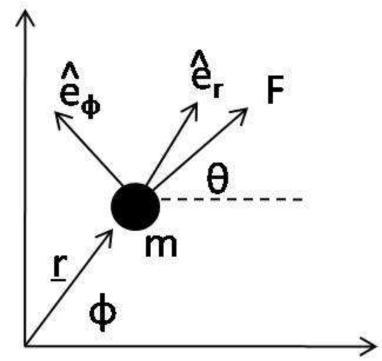

Figure 4: 2-D Particle subject to a force F described by polar coordinates. Figure by MIT OCW.

$$q_1 = r$$
$$q_2 = \phi$$

$$\underline{F} = F_r \hat{e}_r + F_\phi \hat{e}_\phi$$

$$\underline{r} = r(t)\hat{e}_r$$
$$\underline{\dot{r}} = \dot{r}\hat{e}_r + r\dot{\phi}\hat{e}_\phi$$

$$|\underline{v}|^2 = \dot{r}^2 + r^2\dot{\phi}^2$$

$$L = T - V = \frac{1}{2}m(\dot{q}_1^2 + q_1^2\dot{q}_2^2) + 0$$

q_1:

$$\frac{\partial L}{\partial q_1} = m q_1 \dot{q}_2^2$$
$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_1}\right) = m\ddot{q}_1$$

q_2:

$$\frac{\partial L}{\partial q_2} = 0$$

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_2}\right) = \frac{d}{dt}(mq_1^2\dot{q}_2) = m(2q_1\dot{q}_1\dot{q}_2 + q_1^2\ddot{q}_2)$$

$\underline{q_1}(r)$: $Q_1 = F_r$
$Q_2 = F_\phi \cdot r$: moment.

$$m(2\dot{q}_1 q_1 \dot{q}_2 + q_1^2\ddot{q}_2) = F_\phi \cdot q_1$$

$$\boxed{m(2\dot{q}_1\dot{q}_2 + q_1\ddot{q}_2) = F_\phi}$$

$$\boxed{m\ddot{q}_1 - mq_1\ddot{q}_2 = F_r}$$

Example: Falling Stick

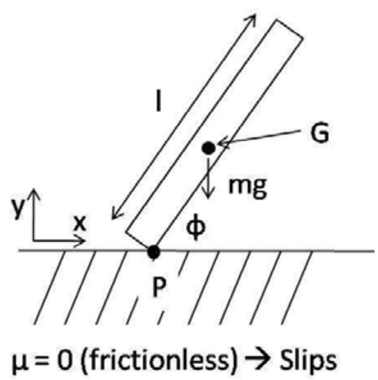

Figure 5: Falling stick. The stick is subject to a gravitational force, mg. The frictionless surface causes the stick to slip. Figure by MIT OCW.

G: Center of Mass
l: length
Constraint: 1 point touching the ground.
2 degrees of freedom

$$q_1 = x_G$$

$$q_2 = \phi$$

Must find L and Q_j. Look for external nonconservative forces that do work.

None. Normal does no work. Gravity is conservaitve.

$$Q_1 = Q_2 = 0$$

Lagrangian

$$L = T - V$$

Rigid bodies: Kinetic energy of translation and rotation

$$T = \frac{1}{2}m(\dot{\underline{r}}_G \cdot \dot{\underline{r}}_G) + \frac{1}{2}I_G(\underline{\omega} \cdot \underline{\omega})$$

$$y_G = \frac{l}{2}\sin\phi$$

$$\dot{y}_G = \frac{l}{2}\cos\phi\dot{\phi}$$

$$\underline{\omega} = \dot{\phi}\hat{k}$$

$$\dot{\underline{r}}_G = \dot{x}_G\hat{\imath} + \dot{y}_G\hat{\jmath} = \dot{x}_G\hat{\imath} + \frac{l}{2}\cos\phi\dot{\phi}\hat{\jmath}$$

$$\dot{\underline{r}}_G \cdot \dot{\underline{r}}_G = \dot{x}_G^2 + \frac{l^2}{4}\cos^2\phi\dot{\phi}^2$$

$$T = \frac{1}{2}\left[\dot{q}_1^2 + \frac{l^2}{4}\cos^2 q_2\dot{q}_2^2\right] + \frac{1}{2}\left(\frac{1}{12}ml^2\right)\dot{q}_2^2$$

See Lecture 16 for the rest of the example.