2.003J/1.053J Dynamics and Control I, Spring 2007
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3/19/2007

Lecture 12

2D Motion of Rigid Bodies: Rolling Cylinder and Rocker Examples

Example: Rolling Cylinder Inside A Fixed Tube

Initial Configuration:

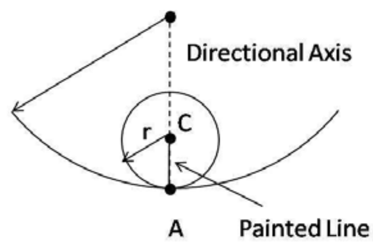

Figure 1: Initial configuration of rolling cylinder inside fixed tube. Figure by MIT OCW.

Derive the equations of motion.
Assume no slip.
Displaced configuration:

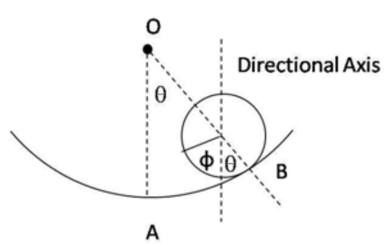

Figure 2: Displaced configuration of rolling cylinder inside fixed tube. Figure by MIT OCW.

Kinematics

May not know everything but at least choose generalized coordinates.
How many generalized coordinates? 3 coordinates initially.
2 constraints.
  1: Rolling on inside of cylinder.
  2: No slip.

Only need 1 generalized coordinate: either ϕ or θ.

We will choose θ.

Recognize angular velocity $\underline{\omega} = -\dot{\phi}\hat{e}_z$.

Must express ϕ in terms of θ.

No-slip condition:

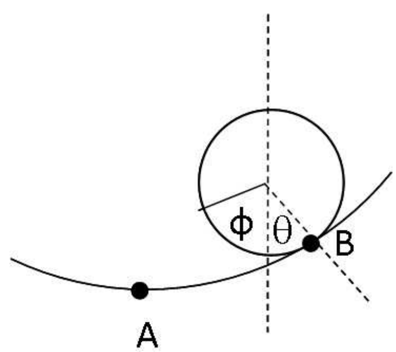

Figure 3: Kinematic diagram of rolling cylinder inside fixed tube. Figure by MIT OCW.

$$R\theta = r(\theta + \phi) \Rightarrow \phi = \frac{(R-r)}{r}\theta;\ \dot{\phi} = \frac{(R-r)}{r}\dot{\theta}$$

Kinetics

3 different methods of solution.
- Angular momentum about B
- Conservation of energy
- Angular momentum about C

Angular Momentum about B

$$\underline{\tau}_B = \frac{d}{dt}\underline{H}_B + \underline{v}_B \times \underline{P}$$

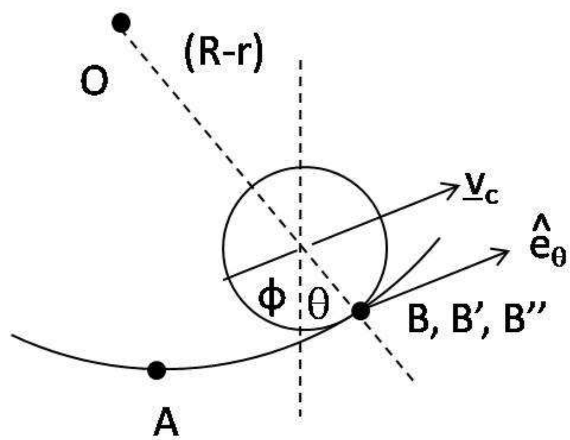

Figure 4: Kinematic diagram of cylinder in fixed tube. Point B is an imaginary particle marking the point of contact. Point B moves. $B^{'}$ on cylinder. $B^{''}$ on tube. If you choose $B^{'}$, at a later point in time $B^{'}$ would have moved away from the contact marker B. Likewise $B^{''}$. Figure by MIT OCW.

$$\underline{v}_B = R\dot{\theta}\hat{e}_\theta \parallel m\underline{v}_c \Rightarrow R\dot{\theta}\hat{e}_\theta = m(R-r)\dot{\theta}\hat{e}_\theta$$

Therefore:

$$\boxed{\underline{v}_B \times \underline{P} = 0}$$

$\underline{H}_B$: $\underline{H}_C + \underline{r}_{BC} \times \underline{P}$
$\underline{r}_{BC} \times \underline{P}$: $\underline{r}_{BC}$ is perpendicular to $\underline{P}$.

$$-rm(R-r)\dot{\theta}\hat{e}_z = \underline{r}_{BC} \times \underline{P}$$

$$\underline{H}_C = I_C\underline{\omega} = \frac{1}{2}mr^2\left(-\frac{(R-r)}{r}\dot{\theta}\hat{e}_z\right)$$

$$\underline{H}_B = \left[-\frac{1}{2}mr(R-r)\dot{\theta} - mr(R-r)\dot{\theta}\right]\hat{e}_z$$

$$\boxed{\underline{H}_B = -\frac{3}{2}mr(R-r)\dot{\theta}\hat{e}_z}$$

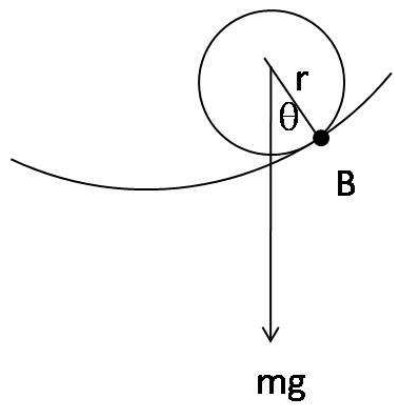

Figure 5: Free body diagram of cylinder. Figure by MIT OCW.

$\underline{\tau}_B$:

$$\boxed{mgr\sin\theta\hat{e}_z}$$

Therefore:

$$mgr\sin\theta = -\frac{3}{2}mr(R-r)\ddot{\theta}$$

$$\boxed{(R-r)\ddot{\theta} + \frac{2}{3}g\sin\theta = 0}$$

<u>Conservation of Energy</u>

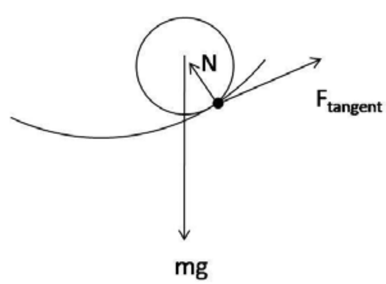

Figure 6: Free body diagram of rolling cylinder in fixed tube. Figure by MIT OCW.

N: normal force
$F_{tangent}$: tangential force
mg: gravity is a conservative force

There is zero velocity at the instant the normal and tangential forces are acting. None of those forces do work, because of the no-slip condition.

$$T + V = \text{Constant}$$

$$\frac{1}{2}mv_c^2 + \frac{1}{2}I_c\omega^2 = T$$

$$\frac{1}{2}[(R-r)\dot{\theta}]^2 + \frac{1}{2}(\frac{1}{2}mr^2)\left[\frac{(R-r)}{r}\dot{\theta}\right]^2 = T$$

$$\boxed{\frac{3}{4}m(R-r)^2\dot{\theta}^2 = T}$$

V:

Define potential energy to be zero at the center of tube (O).

$$\boxed{V = -mg(R-r)\cos\theta}$$

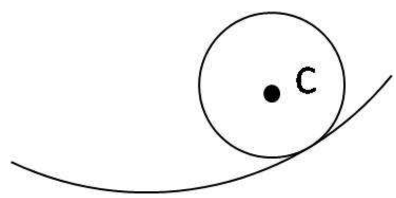

Figure 7: Rolling cylinder in fixed tube. Figure by MIT OCW.

$$\frac{d}{dt}(T+V) = 0\text{: } \frac{3}{4}m(R-r)^2 2\dot{\theta}\ddot{\theta} - mg(R-r)(-\sin\theta\dot{\theta}) = 0$$

$$\boxed{(R-r)\ddot{\theta} + \frac{2}{3}g\sin\theta = 0}$$

Angular Momentum About C

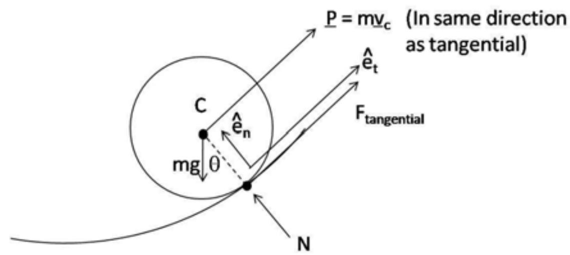

Figure 8: Angular momentum of cylinder about C. Figure by MIT OCW.

Always consider taking angular momentum about the center of mass.

$$v_c \parallel \underline{P} \rightarrow \underline{v}_c \times \underline{P} = 0$$

$$\underline{\tau}_c = \frac{d}{dt}\underline{H}_c$$

$$\underline{\tau}_c = rF\hat{e}_z$$

$$\underline{H}_c\text{:}\ I_c\underline{\omega} = -\frac{1}{2}mr^2\frac{(R-r)}{r}\dot{\theta}\hat{e}_z$$

$$rF = -\frac{1}{2}mr(R-r)\ddot{\theta}$$

We have introduced a force F. Use linear momentum to find expression for F (in F direction).

Apply in direction of F.

$$(F - mg\sin\theta)\hat{e}_t = \frac{d}{dt}m\underline{v}_c\hat{e}_t = \frac{d}{dt}[m(R-r)\dot{\theta}] = m(R-r)\ddot{\theta}\hat{e}_t$$

$$F - mg\sin\theta = m(R-r)\ddot{\theta}$$

There is an error in this analysis.

Where is it?

The error is in $\frac{d}{dt}m\underline{v}_c\hat{e}_t = m(R-r)\ddot{\theta}\hat{e}_t$. $\hat{e}_t$ is changing with respect to time, so we should write $\frac{d}{dt}(mv_c\hat{e}_t)$.

$$
\begin{aligned}
(F - mg\sin\theta)\hat{e}_t &= \frac{d}{dt}(mv_c\hat{e}_t) \\
&= \frac{d}{dt}[m(R-r)\dot{\theta}\hat{e}_t] \\
&= m(R-r)\ddot{\theta}\hat{e}_t + m(R-r)\dot{\theta}\frac{d}{dt}\hat{e}_t \\
&= m(R-r)\ddot{\theta}\hat{e}_t + m(R-r)\ddot{\theta}^2\hat{e}_n
\end{aligned}
$$

$$\frac{d}{dt}\hat{e}_t = \dot{\theta}\hat{e}_n$$

Angular Momentum about C: $F = \frac{1}{2}m(R-r)\ddot{\theta}$
Linear Momentum $\hat{e}_t$: $F - mg\sin\theta = m(R-r)\ddot{\theta}$
Eliminate F from these:

$$\boxed{(R-r)\ddot{\theta} + \frac{2}{3}g\sin\theta = 0}$$

Example: Rocker with Point Mass

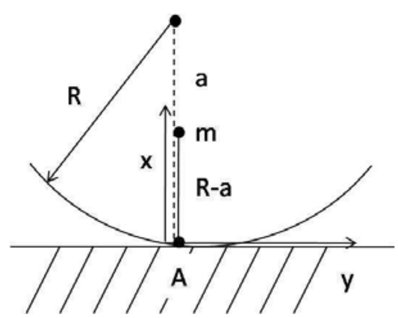

Figure 9: Rocker. All mass at m. No slip. Figure by MIT OCW.

Derive equations of motion.

Kinematics

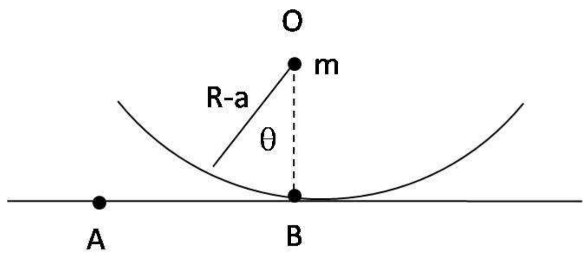

Figure 10: Kinematic diagram of rocker. Figure by MIT OCW.

1 generalized coordinate θ.

$$x = R\theta - a\sin\theta$$
$$y = R - a\cos\theta$$

$$\dot{x} = R\dot{\theta} - a\cos\theta\dot{\theta}$$
$$\dot{y} = a\sin\theta\dot{\theta}$$

Because of no slip, $(R-a)\theta = \overline{AB}$.

Kinetics

Conservation of Energy

Usable, similar argument to one in rolling cylinder example based on no-slip condition.

Angular Momentum about B

$$\tau_B = \underline{\dot{H}}_B + \underline{v}_B \times \underline{P}$$

$\underline{v}_B \times \underline{P}$: This term is not zero.

B always lies below point O.

$$\underline{v}_B = R\dot{\theta}\hat{e}_x$$
$$\underline{P} = m(R\dot{\theta} - a\cos\theta\dot{\theta})\hat{e}_x + m(a\sin\theta\dot{\theta})\hat{e}_y$$

Solution to Mass Particle in Rocking Chair Example

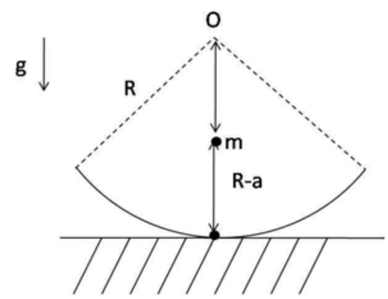

Figure 11: Mass particle in rocking chair. Assume pure rolling. Figure by MIT OCW.

Kinematics

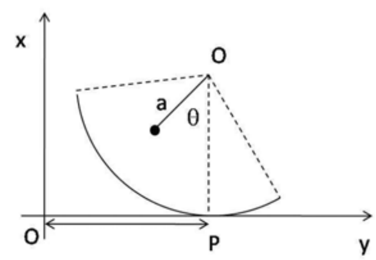

Figure 12: Kinematic diagram of mass particle in rocking chair. Figure by MIT OCW.

Center of mass (particle) coordinates: x, y

Due to rolling: $|OP| = R\theta$

$$x = R\theta - a\sin\theta \Rightarrow \dot{x} = R\dot{\theta} - a\dot{\theta}\cos\theta$$
$$y = R - a\cos\theta \Rightarrow \dot{y} = a\sin\theta\dot{\theta}$$

$$\ddot{x} = R\ddot{\theta} + a\dot{\theta}^2\sin\theta - a\ddot{\theta}\cos\theta$$
$$\ddot{y} = a\ddot{\theta}\sin\theta + a\cos\theta\dot{\theta}^2$$

1 generalized coordinate θ.

Kinetics (Forces)

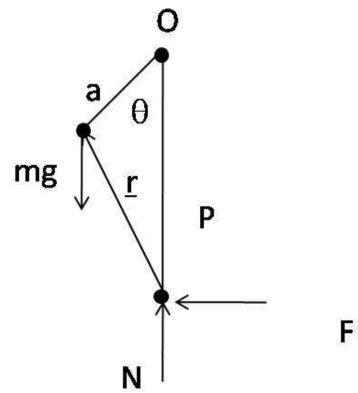

Figure 13: Free body diagram of mass particle and rocking chair. Figure by MIT OCW.

Angular Momentum Principle: Moments Taken around P

To eliminate $\underline{N}$, $\underline{F}$ take moments about point P that moves such that it is always under O. $\underline{v}_P = R\dot{\theta}\hat{\imath}$.

$$\sum \underline{\tau}_P^{ext} = \frac{d}{dt}\underline{H}_P + \underline{v}_P \times \underline{P}$$

$$\sum \underline{\tau}_P^{ext} = mga\sin\theta\hat{e}_z$$

$$\underline{P} = m\dot{x}\hat{\imath} + m\dot{y}\hat{\jmath}$$

$$\underline{v}_P \times \underline{P} = mRa\dot{\theta}^2\sin\theta\hat{e}_z$$

$$\underline{H}_P = \underline{H}_C + \underline{r} \times \underline{P} = \underline{r} \times m(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})$$

$$\underline{r} = -a\sin\theta\underline{L} + (R - a\cos\theta)\hat{\jmath}$$

$$\underline{H}_P = [-m\dot{x}(R - a\cos\theta) - m\dot{y}a\sin\theta]\hat{e}_z$$

$$\frac{d}{dt}\underline{H}_P = -[(R^2 - 2Ra\cos\theta + a^2)\ddot{\theta} + 2a\dot{\theta}^2\sin\theta]\hat{e}_z$$

Use the above to substitute for the terms in $\sum \underline{\tau}_P^{ext} = \frac{d}{dt}\underline{H}_P + \underline{v}_P \times \underline{P}$.

$$\boxed{(R^2 - 2Ra\cos\theta + a^2)\ddot{\theta} + Ra\dot{\theta}^2\sin\theta + ga\sin\theta = 0}$$

Second order nonlinear: Given $\theta(0)$ and $\dot{\theta}(0)$ may integrate (Matlab) to find time history for $t > 0$.

Conservation of Energy

Could we get this equation from other means? Is system conservative? Yes, $\underline{N}$ and $\underline{F}$ do *no* work.
Then:

$$T + V = \text{ Constant}$$

$$\begin{aligned} T &= \frac{1}{2}m(\dot{x}^2 + \dot{y}^2) = \frac{1}{2}m[(R\dot{\theta} - a\dot{\theta}\cos\theta)^2 + a^2\dot{\theta}^2\sin^2\theta] \\ &= \frac{1}{2}m(R^2 - 2Ra\cos\theta + a^2)\dot{\theta}^2 \end{aligned}$$

$$V = mgy = mg(R - a\cos\theta)$$

$$mg(R - a\cos\theta) + \frac{1}{2}m(R^2 - 2Ra\cos\theta + a^2)\dot{\theta}^2 = \text{ Constant}$$

Taking the time derivative:

$$mga\dot{\theta}\sin\theta + m\dot{\theta}\ddot{\theta}(R^2 - 2Ra\cos\theta + a^2) + mRa\sin\theta\dot{\theta}^2\dot{\theta} = 0$$

$$\boxed{(R^2 - 2Ra\cos\theta + a^2)\ddot{\theta} + Ra\sin\theta\dot{\theta}^2 + ga\sin\theta = 0}$$

Angular Momentum Principle: Moments Taken About Center of Mass

Forces:

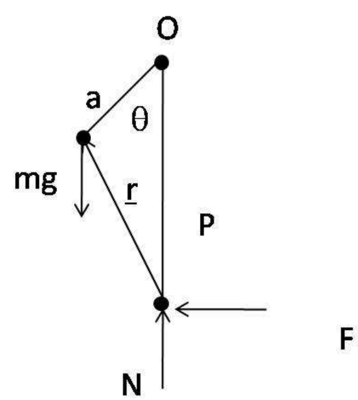

Figure 14: Free body diagram of mass particle and rocking chair. Figure by MIT OCW.

$$\sum \underline{F}^{ext} = \frac{d}{dt}[m(\dot{x}\hat{\imath} + \dot{y}\hat{\jmath})]$$
$$-F = m\ddot{x} = m(R\ddot{\theta} + a\dot{\theta}^2 \sin\theta - a\ddot{\theta}\cos\theta) \quad (1)$$
$$N - mg = m\ddot{y} = m(a\ddot{\theta}\sin\theta + a\dot{\theta}^2\cos\theta) \quad (2)$$

Need to relate $\underline{N}$ and $\underline{F}$.

$$\sum \underline{\tau}_P^{ext} = \frac{d}{dt}\underline{H}_C$$
$$\underline{H}_C = 0$$

$$\sum \underline{\tau}_P^{ext} = 0 \Rightarrow N \cdot a\sin\theta = F \cdot (R - a\cos\theta) \quad (3)$$

$$N = mg + ma\ddot{\theta}\sin\theta + ma\dot{\theta}^2\cos\theta$$
$$Na\sin\theta = mga\sin\theta + ma^2\ddot{\theta}\sin^2\theta + ma^2\dot{\theta}^2\cos\theta\sin\theta$$

From (1):

$$F = -m(R - a\cos\theta)\ddot{\theta} - ma\dot{\theta}^2\sin\theta$$
$$F(R - a\cos\theta) = -m(R - a\cos\theta)^2\ddot{\theta} - ma\dot{\theta}^2\sin\theta(R - a\cos\theta)$$

$$-m(R - a\cos\theta)^2\ddot{\theta} - ma\dot{\theta}^2\sin\theta R = mga\sin\theta + ma^2\ddot{\theta}\sin^2\theta$$
$$m(R^2 + a^2\cos^2\theta - 2Ra\cos\theta)\ddot{\theta} + ma\dot{\theta}^2 R\sin\theta + mga\sin\theta + ma^2\ddot{\theta}\sin^2\theta = 0$$
$$\boxed{(R^2 - 2Ra\cos\theta + a^2)\ddot{\theta} + ma\dot{\theta}^2 R\sin\theta + mga\sin\theta = 0}$$

Analysis of Equation of Motion

Equilibrium: $\ddot{\theta} = \dot{\theta} = 0$

$$mga\sin\theta = 0 \Rightarrow \theta = 0$$

How do we describe the oscillatory motion about $\theta = 0$?

For θ, $\dot{\theta}$, $\ddot{\theta}$ small about $\theta = 0$ we may approximate:

$$\sin\theta \approx \theta$$
$$\cos\theta \approx 0$$

$$m(R^2 - 2Ra + a^2)\ddot{\theta} + mga\theta = 0$$
$$(R - a)^2\ddot{\theta} + ga\theta = 0$$

$$\ddot{\theta} + \frac{ga}{(R-a)^2}\theta = 0$$

$$[\ddot{x} + \frac{k}{m}x = 0]$$

Natural Frequency:

$$\omega = \frac{2\pi}{T} = \sqrt{\frac{ga}{(R-a)^2}}$$

$$T = 2\pi\sqrt{\frac{(R-a)^2}{ga}}$$

Use solutions of the form Ae^{mt}:

$$\Rightarrow m^2 + \frac{ga}{(R-a)^2} = 0$$

$R \approx 30$ cm, $a \approx 15$ cm, $T \approx 0.785$.
$m = \pm i\sqrt{\frac{ga}{(R-a)^2}}$

$$\theta \approx Ae^{it\sqrt{\frac{ga}{(R-a)^2}}} + Be^{-it\sqrt{\frac{ga}{(R-a)^2}}} = C\cos\left(t\sqrt{\frac{ga}{(R-a)^2}}\right) + D\sin\left(t\sqrt{\frac{ga}{(R-a)^2}}\right)$$