2.003J/1.053J Dynamics and Control I, Spring 2007
Professor Thomas Peacock
2/12/2007

Lecture 2

Work-Energy Principle

Dynamics of a Single Particle (Review) (continued)

Reading: Williams 4–1, 4–2, 4–3 (Momentum Principles) - End of lecture last time

Work-Energy Principle

This material is covered in Williams 5–1, 5–2, and 5–3.

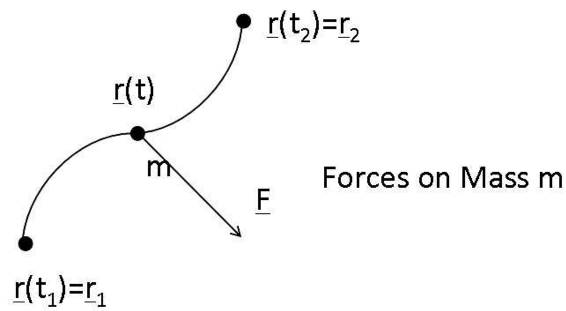

Figure 1: Single particle moving. m moves along $r(t)$ and experiences a force, F. Figure by MIT OCW.

W_{12} =work done going from $1 \rightarrow 2$

$$\begin{aligned}\text{Integral over sum} &= \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}\cdot d\underline{r} = \text{“Work done by force F(t)” (in going from } \underline{r}_1 \text{ to } \underline{r}_2)\\ &= \int_{t_1}^{t_2} \underline{F}\cdot \underline{v}dt \text{ (Integration over time)}\\ &= \int_{t_1}^{t_2} \frac{d}{dt}(m\underline{v})\cdot \underline{v}dt \text{ where } m \text{ is constant}\end{aligned}$$

$$= \int_{t_1}^{t_2} \frac{d}{dt}\left[\frac{1}{2}m\underline{v}\cdot\underline{v}\right]dt$$
$$= \frac{1}{2}m\underline{v}\cdot\underline{v}|_{t_1}^{t_2} = \frac{1}{2}mv_2^2 - \frac{1}{2}mv_1^2$$

Define Kinetic Energy $T = \frac{1}{2}m|\underline{v}|^2$
$\Rightarrow W_{12} = T_2 - T_1 =$ (Work done is the change in the kinetic energy)

Consider the case:

$$\underline{F} = \frac{-\partial V}{\partial \underline{r}}$$

For example,

$$\underline{F} = (\frac{\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath})$$

$$\Rightarrow W_{12} = \int_{\underline{r}_1}^{\underline{r}_2} \underline{F}\cdot d\underline{r} = \int_{\underline{r}_1}^{\underline{r}_2} \frac{-\partial V}{\partial \underline{r}}\cdot d\underline{r} = V_1 - V_2 \text{(Potential Energy)}$$

Thus, in this special case:

$$T_2 - T_1 = V_1 - V_2$$

Or Conservation of Mechanical Energy

$$T_2 + V_2 = T_1 + V_1 = \text{ Constant}$$

Forces which can be written $\underline{F} = \frac{-\partial V}{\partial \underline{r}}$ (i.e. as the gradient of a potential) are called "Conservative Forces."
Examples:

Gravity

$$\underline{F} = -mg\hat{\jmath}$$
$$V = mgy \Rightarrow \frac{-\partial V}{\partial \underline{r}} = \frac{-\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath} = -mg\hat{\jmath}$$

Linear Spring

$$\underline{F} = -kx\hat{\imath}$$
$$V = \frac{1}{2}kx^2 \Rightarrow \frac{-\partial V}{\partial \underline{r}} = \frac{-\partial V}{\partial x}\hat{\imath} - \frac{\partial V}{\partial y}\hat{\jmath} = -kx\hat{\imath}$$

Linear Dashpot: Frictional force that opposes the direciton of motion. Depends on velocity. EX: Slows motion down, door opener.

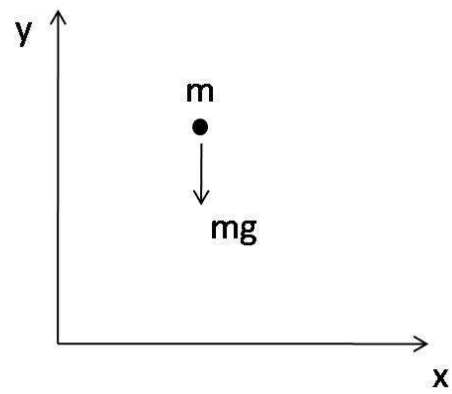

Figure 2: Free body diagram. The only force acting on the mass is the force due to gravity mg. Figure by MIT OCW.

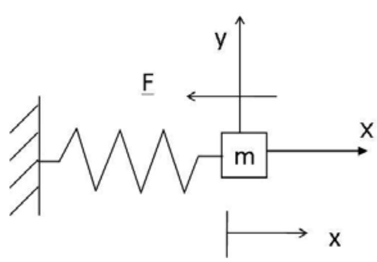

Figure 3: Mass attached to a spring. Figure by MIT OCW.

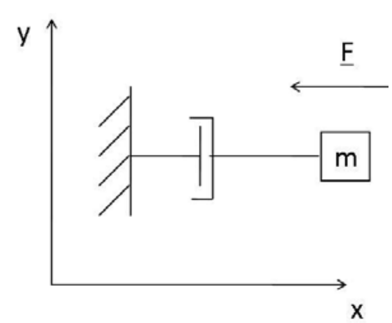

Figure 4: Mass attached to linear dashpot. Figure by MIT OCW.

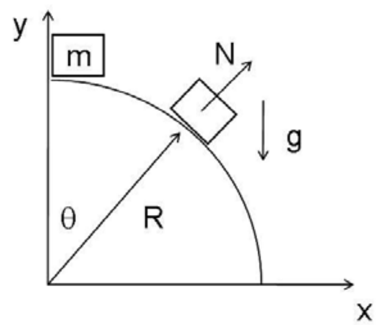

Figure 5: Vehicle falling on a curve. The vehicle with mass m starts at the top of the hill and falls along curve. Figure by MIT OCW.

$$\underline{F} = c\dot{x}\hat{\imath} \neq \frac{-\partial V}{\partial \underline{r}}$$

Not conservative. m experiences same force as long as velocity is same. Position independent. The constant c meanst this is linear dashpot behavior.

If all forces are potentiall or the ones that are not do work (e.g. always perpendicular to motion) we call the particle motion "conservative" because mechanical energy (kinetic + potential) is conserved.

Discussion:

$$\underline{F} = \underline{F}_{conservative} + \underline{F}_{non-conservative}$$

Using conservation of mechanical energy can be useful

1. Scalar Equation
2. Need not account for forces that do no work

Example 1: Vehicle Falling on a Curve

A small vehicle is released from rest at the top of a circular path. Determine the angle θ_0 to the position where the vehicle leaves the path and becomes a projectile.

(Neglect friction and treat vehicle as a particle.)

Problem relates to driving a car down a hill.

Whenever you solve a problem, make sure that you draw a good diagram.

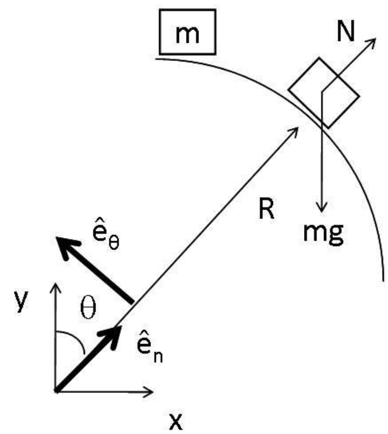

Figure 6: Free body diagram of vehicle falling along curve. The two forces on the vehicle are the normal force, N, and the force due to gravity mg. Figure by MIT OCW.

When does this car become free? When N is zero, the vehicle has left the surface. How does the point of departure (θ) depend on R or m?

Free Body Diagram (FBD)

$$\hat{e}_n\text{: Unit vector normal } \hat{e}_t\text{: Unit vector tangential}$$

$$\underline{r} = R\sin\theta\hat{\imath} + R\cos\theta\hat{\jmath}$$

$$\underline{\dot{r}} = R\dot{\theta}(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) = R\dot{\theta}\hat{e}_t$$

$$\underline{\dot{p}} = m\underline{\dot{v}} = mR\ddot{\theta}(\cos\theta\hat{\imath} - \sin\theta\hat{\jmath}) + mR\dot{\theta}^2(-\sin\theta\hat{\imath} - \cos\theta\hat{\jmath}) = mR\ddot{\theta}\hat{e}_t - mR\dot{\theta}^2\hat{e}_n$$

$$\sum \underline{F} = -mg\hat{\jmath} + N\hat{e}_n = (N - mg\cos\theta)\hat{e}_n + mg\cos\theta\hat{e}_t$$

$$\text{From Newton II: } \sum \underline{F} = \underline{\dot{p}}$$

$$N - mg\cos\theta = -mR\dot{\theta}^2 \qquad \text{Normal} \tag{1}$$

$$mg\sin\theta = mR\ddot{\theta} \qquad \text{Tangential} \tag{2}$$

We have:

$$-mg\cos\theta + N = -mR\dot{\theta}^2$$

Vehicle will leave track when $N = 0$

$$-mg\cos\theta_0 = -mR\dot{\theta}^2$$
$$R\dot{\theta}^2 = g\cos\theta_0$$

(This is true only when $N = 0$. We want to know $\dot{\theta}^2 = f(\theta)$?)

Could use Equation (2) to get information about $\dot{\theta}^2$ by integration, but instead we will use conservation of energy to get a relation $\dot{\theta}^2 = f(\theta)$
Forces:
- Gravity - Conservative
- No Friction
- Normal Force: N is always $\perp$ to motion. Does no work.

We have a conservative system.
Apply conservation of mechanical energy.

$$T_{TOP} + V_{TOP} = T_\theta + V_\theta$$
$$|\underline{v}| = R\dot{\theta}$$

$$0 + mgR = \frac{1}{2}mR^2\dot{\theta}^2 + mgR\cos\theta$$
$$\dot{\theta}^2 = \frac{2g}{R}(1-\cos\theta)$$

Putting into (1) and letting $N = 0$, $R\dot{\theta}^2 = R\frac{2g}{R}(1-\cos\theta) = g\cos\theta_0$.

$$2(1-\cos\theta_0) = \cos\theta_0$$
$$\cos\theta_0 = \frac{2}{3}$$
$$\theta_0 = 48.2^\circ$$

Note: This solution is independent of R and m.

Linear momentum principle used to solve problem.
Next time, use angular momentum principle to solve a problem.