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2.002 MECHANICS AND MATERIALS II
QUIZ I
SOLUTIONS

Distributed: Wednesday, March 17, 2004

This quiz consists of four (4) questions. A brief summary of each question's content and associated points is given below:

1. **(10 points)** This is the credit for your (up to) two (2) pages of self-prepared notes. **Please be sure to put your name on each sheet**, and hand it in with the test booklet. You are already done with this one!
2. **(20 points)** A lab-based question.
3. **(40 points)** A multi-part question about a linear elastic boundary value problem.
4. **(30 points)** A 'design for yield' question.

Note: you are encouraged to write out

- your understanding of the problem, and
- your understanding of "what to do in order to solve the problem", even if you find yourself having "algebraic/numerical difficulties" in actually doing so: in short,
- let me see what you are thinking, instead of just what you happen to write down....

The last page of the quiz contains "useful" information. Please refer to this page for equations, etc., as needed.

If you have any questions about the quiz, please ask for clarification.

Good luck!

Problem 1 (10 points)

Attach your self-prepared 2-sheet (4-page) notes/outline to the quiz booklet.

Be sure your name is on each sheet.

Problem 2 (20 points)

(Lab-Based Problem)

In your own words, describe the following terms used in the description of linear elastic stress concentration, and briefly illustrate (with schematic figures, simple equations, etc.) how these features are used, measured, or are otherwise identified:

- (5 points) **stress concentration factor**

  A stress concentration factor, K_t, can be defined as the ratio of the [peak] value of a stress component,σ_{local}, at a highly-stressed location (e.g., a notch root) to a nominal, or far-field value of a stress component, σ_{nom}, that is [typically] readily associated with overall loading:

$$K_t \equiv \frac{\sigma_{\text{local}}}{\sigma_{\text{nom}}} \geq 1.$$

- (5 points) **St. Venant's principle** St. Venant's principle states that the perturbation in a nominally homogeneous stress state introduced by a [geometric/material] heterogeneity (e.g., a hole, notch, cut-out, or reinforcement) of characteristic linear dimension "ℓ" decays rapidly with distance from the heterogeneity. In practice, the effects of the perturbation in the stress field are negligible for distances greater than $\sim 3\ell$.

(10 points) Describe "best practice" in the location of a resistance strain gauge to measure stress concentration at the root of a through-thickness notch in a planar body subjected to in-plane loading. (1 page, maximum).

The key ideas here are to note that (a) a strain gauge records an *average* value of strain parallel to its direction, with the region sampled being that beneath the gauge, and (b), in the presence of strong stress concentration, there are steep gradients in the values of in-plane strain (and stress) in the immediate vicinity of the notch root. Thus, any attempt to use a laterally-mounted strain gauge in order to estimate the strain concentration "at" the root of the notch will be hindered by the inherent averaging-in of lower [than peak] strain in the region under the gauge.

Conversely, viewing the body as having a small but finite thickness, t, at its notch root, the variation in strain along the notch is generally much less rapid than its variation "radially" away from the notch surface. Thus, a strain gauge mounted on the t-thick surface of the notch root provides a much more reliable estimate of the peak strain at the notch than can be obtained from a similar gauge mounted laterally on the faces of the body.

Problem 3 (40 points)

A large isotropic linear elastic body contains a long, embedded fiber, of radius a, extending parallel to the x_3 axis (see Fig. 1, below). The fiber is made of a different material than the surrounding matrix; elastic moduli of the fiber *greatly exceed* those of the surrounding matrix. Thus, as an approximation, we can treat the fiber as a *rigid, non-deforming body.*

Under a certain loading of the fiber-containing matrix, the spatial variation of the components u_i of the displacement vector $\mathbf{u}$ in the matrix region is described by

$$\begin{aligned} u_1(x_1, x_2, x_3) &= 0 \\ u_2(x_1, x_2, x_3) &= 0 \\ u_3(x_1, x_2, x_3) &= \gamma\, x_2 \left(1 - \frac{a^2}{x_1^2 + x_2^2}\right), \end{aligned}$$

for a dimensionless constant "γ" satisfying $|\gamma| \ll 1$. (N.B.: These expression apply only in the matrix region exterior to the rigid fiber; that is, at locations satisfying $x_1^2 + x_2^2 \equiv r^2 \geq a^2$.

- (**5 points**) **Evaluate the spatial dependence of all components of the strain tensor,** ϵ_{ij}

$$\begin{aligned} \epsilon_{11} &\equiv \frac{\partial u_1}{\partial x_1} = 0 \\ \epsilon_{22} &\equiv \frac{\partial u_2}{\partial x_2} = 0 \\ \epsilon_{33} &\equiv \frac{\partial u_3}{\partial x_3} = 0 \\ \epsilon_{12} = \epsilon_{21} &\equiv \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) = 0 \\ \epsilon_{13} = \epsilon_{31} &\equiv \frac{1}{2}\left(\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right) = \frac{x_1 x_2 \gamma a^2}{(x_1^2 + x_2^2)^2} \\ \epsilon_{23} = \epsilon_{32} &\equiv \frac{1}{2}\left(\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right) = \frac{\gamma}{2}\left[1 - \frac{a^2}{x_1^2 + x_2^2} + \frac{2a^2 x_2^2}{(x_1^2 + x_2^2)^2}\right] \end{aligned}$$

- (**5 points**)Based on the strain components calculated above, **evaluate the spatial dependence of all components of the stress tensor,** σ_{ij}.

$$\sigma_{ij} = \frac{E}{(1+\nu)}\left[\epsilon_{ij} + \frac{\nu}{(1-2\nu)}\left(\sum_{m=1}^{3} \epsilon_{mm}\right)\delta_{ij}\right].$$

$$\sigma_{11} = \sigma_{22} = \sigma_{33} = \sigma_{12} = \sigma_{21} = 0;$$

$$\sigma_{13} = \sigma_{31} = \frac{E}{1+\nu}\,\epsilon_{13} = 2G\epsilon_{13} = \frac{2x_1x_2G\gamma a^2}{(x_1^2+x_2^2)^2}$$

$$\sigma_{23} = \sigma_{32} = \frac{E}{1+\nu}\,\epsilon_{23} = 2G\epsilon_{23} = G\gamma\,\left[1 - \frac{a^2}{x_1^2+x_2^2} + \frac{2a^2x_2^2}{(x_1^2+x_2^2)^2}\right]$$

(5 points) Describe the state of stress "far" from the fiber. Distance from the fiber can be measured by radius $r \equiv \sqrt{x_1^2+x_2^2}$; when $r/a \gg 1$ (or, equivalently, when $a/r \ll 1$, the point is far from the outer boundary of the fiber. By introducing the distance variable r into the spatial dependence of the two non-zero stress components, we see that

$$\sigma_{23} = \sigma_{32} = G\gamma\,\left[1 - \frac{a^2}{r^2} + \frac{2a^2}{r^2}\frac{x_2^2}{r^2}\right] \rightarrow G\gamma$$

$$\sigma_{13} = \sigma_{31} = 2G\epsilon_{13} = G\gamma\,\frac{2x_1x_2}{r^2}\frac{a^2}{r^2} \rightarrow 0$$

The 'far' limits apply because $a/r \ll 1$ (and even more so for $(a/r)^2$), while $x_1x_2/r^2 = \sin\theta\cos\theta$ remains bounded for large r/a.

- (**10 points**) **Show that the stress tensor components calculated above satisfy the equilibrium equations.** A bit of reflection concerning the equilibrium equations

$$\sum_{j=1}^{3}\frac{\partial\sigma_{ij}}{\partial x_j} = 0_i$$

will show that the only relevant equation is for equilibrium in the x_3 direction ($i = 3$):

$$\frac{\partial\sigma_{31}}{\partial x_1} + \frac{\partial\sigma_{32}}{\partial x_2} = 0.$$

The non-zero stress component σ_{13} (which appears, differentiated with respect to x_3 in the x_1-direction equilibrium equation), *does not depend on* x_3; likewise, the stress component σ_{23} appears in the x_2-direction equilibrium equation, but only when it is differentiated with respect to x_3, on which it does not depend. Thus, only the x_3 component remains. Using the previous expressions,

$$\begin{aligned}\frac{\partial\sigma_{31}}{\partial x_1} &= G\gamma\,\frac{\partial\left[2x_1x_2a^2r^{-4}\right]}{\partial x_1}\\ &= G\gamma\left[2a^2x_2r^{-4} + (2a^2x_1x_2)(-4r^{-5}x_1/r)\right]\\ &= G\gamma\,2a^2x_2r^{-4}\left[1 - 4x_1^2/r^2\right];\end{aligned}$$

$$
\begin{aligned}
\frac{\partial \sigma_{32}}{\partial x_2} &= G\gamma \, \frac{\partial[1 - a^2/r^2 + 2a^2 x_2^2/r^4]}{\partial x_2} \\
&= G\gamma \left[2a^2 x_2 r^{-4} + 2a^2 \left(2x_2 r^{-4} - 4x_2^3 r^{-6}\right)\right] \\
&= G\gamma \, 2a^2 x_2 r^{-4} \left[3 - 4x_2^2/r^2\right]
\end{aligned}
$$

In writing these two evaluations, use was made of the chain rule and the relation $\partial r^2/\partial x_1 = 2r\partial r/\partial x_1 = 2x_1 \Rightarrow \partial r/\partial x_1 = x_1/r$, along with the related identity $\partial r/\partial x_2 = x_2/r$.

We can now add the two expressions in the x_3-component of the equilibrium equation:

$$
\begin{aligned}
0 &= \frac{\partial \sigma_{31}}{\partial x_1} + \frac{\partial \sigma_{32}}{\partial x_2} \\
&= G\gamma \, 2a^2 x_2 r^{-4} \left[(1 - 4\frac{x_1^2}{r^2}) + (3 - 4\frac{x_2^2}{r^2})\right] \\
&= G\gamma \, 2a^2 x_2 r^{-4} \left[(1 + 3) - 4\frac{(x_1^2 + x_2^2)}{r^2})\right] \\
&= 0.
\end{aligned}
$$

The x_3 component of static equilibrium is indeed satisfied.

- (**5 points**) A particular point "P" on the interface between the fiber and the matrix is located at in-plane position $\mathbf{x}^P = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 = a\cos\theta\mathbf{e}_1 + a\sin\theta\mathbf{e}_2$

  – **evaluate the components, $\{n_i\}$, of the unit normal vector, n, pointing from the surface of the matrix at P toward the interior of the fiber.** This unit vector is just $\mathbf{n} = -\left(\cos\theta\,\mathbf{e}_1 + \sin\theta\,\mathbf{e}_2\right);$

$$
\left\{\begin{array}{c} n_1 \\ n_2 \\ n_3 \end{array}\right\} = \left\{\begin{array}{c} -\cos\theta \\ -\sin\theta \\ 0 \end{array}\right\}.
$$

  – **evaluate the components, $\{t_i\}$, of the traction vector acting on the surface of the matrix at P.**

$$
\left\{\begin{array}{c} t_1 \\ t_2 \\ t_3 \end{array}\right\} = \left[\begin{array}{ccc} 0 & 0 & \sigma_{13} \\ 0 & 0 & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & 0 \end{array}\right] \left\{\begin{array}{c} -\cos\theta \\ -\sin\theta \\ 0 \end{array}\right\} = -\left\{\begin{array}{c} 0 \\ 0 \\ \sigma_{31}\cos\theta + \sigma_{32}\sin\theta \end{array}\right\}.
$$

- **(10 points) Use your physical understanding of this problem to define and evaluate an appropriate <u>stress concentration factor</u>.** Examination of the peak values of the stress components shows that the peak stress occurs for component σ_{32} at locations just on the top/bottom of the fiber, at $x_1 = 0$ and $x_2 = \pm a$. At those points, $\sigma_{32}^{\text{local}} = 2G\gamma = 2\sigma_{32}^{\text{remote}}$. Thus, the stress concentration factor can be defined as

$$K_t = \frac{\sigma_{32}(x_1 = 0; x_2 = \pm a)}{\sigma_{32}(r \to \infty)} = 2.$$

Useful information: $2G(1+\nu) = E$, where G is the isotropic linear elastic shear modulus, E is the Young's modulus, and ν is Poisson's ratio.

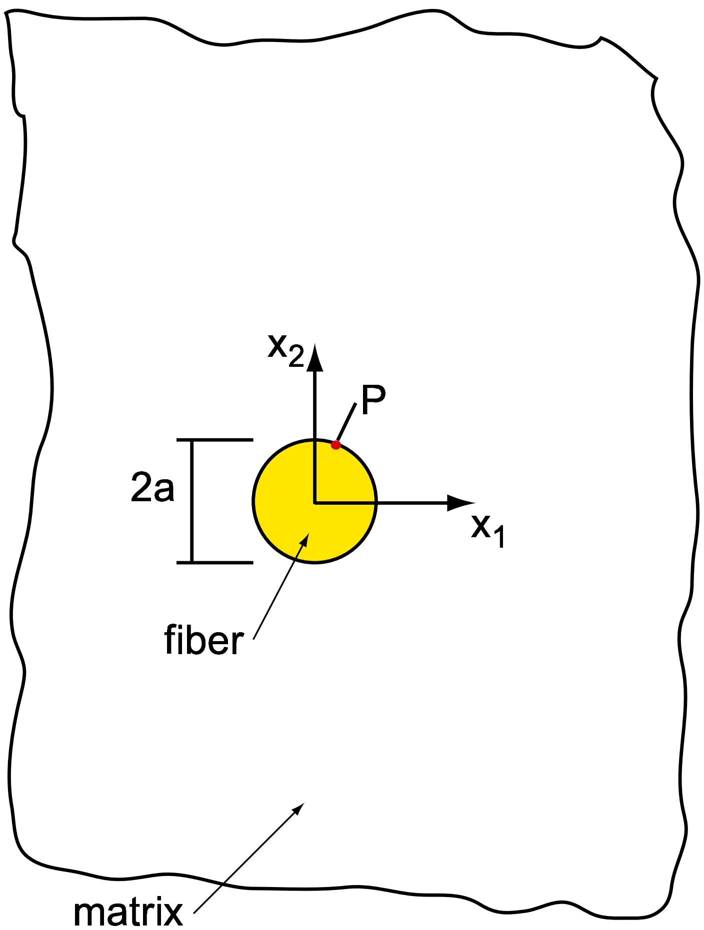

Figure 1: Cross-section of large linear elastic body ("the matrix") containing a long fiber of diameter $2a$ parallel to the x_3-axis. The elastic moduli of the fiber are *much larger* than those of the matrix, so the fiber is idealized as being rigid, or elastically non-deformable, in comparison.

Problem 4 (30 points)

A closed-ended thin-walled circular cylindrical pressure vessel has wall thickness $t = 5\,mm$ and mean radius $\bar{R} = 50\,mm$. Material properties for the tube include $E = 208\,GPa$, $\nu = 0.30$, and $\sigma_y = 500\,MPa$.

- (**10 points**) At zero internal pressure, the long closed tube is subjected to an unknown axial tensile force, N, which causes an axially-oriented strain gauge mounted on the outer diameter of the tube to register a strain of magnitude 0.0010. **Evaluate N.**
Solution: The stress state in the tube wall, due to axial force N alone, over the tube cross-sectional area $A \doteq 2\pi\bar{R}t$, is uniaxial, of magnitude $\sigma_{zz} = N/A$; all other cylindrical stress components are zero: $\sigma_{rr} = \sigma_{\theta\theta} = \sigma_{r\theta} = \sigma_{rz} = \sigma_{\theta z} = 0$. Under uniaxial stress, the axial stress and strain components are related by

$$\sigma_{zz} = E\epsilon_{zz} = \left(208 \times 10^3\,MPa\right) \times 10^{-3} = 208\,MPa \equiv \sigma_{zz}^{(N)}.$$

The axial load/stress relation is

$$N = \left(2\pi\bar{R}t\right)\,\sigma_{zz}^{(N)} = 2\pi\,(50\,mm)(5mm) \times (208\,MPa) \times \frac{N/mm^2}{MPa} = 326.7\,kN$$

- (**15 points**) With the load N held constant, gas is introduced into the cylinder, causing the internal pressure, p, to increase. **Evaluate the maximum pressure that can be applied without initiating plastic deformation in the wall of the tube.**
Solution: The internal pressure of magnitude p creates both an axial and a hoop stress stress component in a closed-ended, thin-walled cylinder:

$$\sigma_{\theta\theta} = \frac{p\bar{R}}{t}; \quad \sigma_{zz} = \frac{p\bar{R}}{2t}; \quad \sigma_{rr} \doteq 0.$$

These stresses must be superposed with those due to the axial force, N; The resulting non-zero stress components are

$$\sigma_{\theta\theta} = \frac{p\bar{R}}{t} \equiv \Sigma;$$

$$\sigma_{zz} = \frac{p\bar{R}}{2t} + \sigma_{zz}^{(N)} = \Sigma/2 + \sigma_{zz}^{(N)},$$

where $\sigma_{zz}^{(N)} = 208\,MPa$ is that due to the axial load, as noted above. The initial yield condition can be expressed in terms of the Mises equivalent tensile stress, $\bar{\sigma}$, as

$$\bar{\sigma} = \sigma_y, \quad \text{or}$$

$$
\begin{aligned}
\sigma_y^2 &= \bar{\sigma}^2 = \frac{1}{2}[(\sigma_{zz} - \sigma_{\theta\theta})^2 + \sigma_{zz}^2 + \sigma_{\theta\theta}^2] \\
&= \sigma_{zz}^2 + \sigma_{\theta\theta}^2 - \sigma_{zz}\sigma_{\theta\theta} \\
&= \left(\frac{\Sigma}{2} + \sigma_{zz}^{(N)}\right)^2 + (\Sigma)^2 - \Sigma\left(\frac{\Sigma}{2} + \sigma_{zz}^{(N)}\right) \\
&= \Sigma^2\left(1 + \frac{1}{4} - \frac{1}{2}\right) + \Sigma\sigma_{zz}^{(N)} + \left(\sigma_{zz}^{(N)}\right)^2 - \Sigma\sigma_{zz}^{(N)} \\
\sigma_y^2 &= \frac{3}{4}\Sigma^2 + \left(\sigma_{zz}^{(N)}\right)^2
\end{aligned}
$$

This can be solved as a quadratic expression for Σ:

$$
\Sigma^2 = \frac{4}{3}\left[\sigma_y^2 - (\sigma_{zz}^{(N)})^2\right];
$$

$$
\Sigma = \frac{2}{\sqrt{3}}\sqrt{\sigma_y^2 - (\sigma_{zz}^{(N)})^2} = \frac{2}{\sqrt{3}}\sqrt{(500\,MPa)^2 - (208\,MPa)^2} = 525\,MPa
$$

Recalling the definition of $\Sigma = p\bar{R}/t$, the corresponding pressure is $p = \Sigma t/\bar{R} = 525\,MPa \times (5\,mm/50\,mm) = 52.5\,MPa$.

- (**5 points**) With the load N applied, along with the pressure level determined above, **what values will the strain gauge read?**

Using the linear elastic stress-strain equations, the axial strain is related to the stress by

$$
\epsilon_{zz} = \frac{1}{E}\left[\sigma_{zz} - \nu\left(\sigma_{rr} + \sigma_{\theta\theta}\right)\right].
$$

In our case, $\sigma_{rr} = 0$, $\sigma_{\theta\theta} = p\bar{R}/t = \Sigma = 525\,MPa$, and $\sigma_{zz} = \sigma_{zz}^{(N)} + p\bar{R}/2t = 208\,MPa + 262.5\,MPa = 470.5\,MPa$. Thus

$$
\epsilon_{zz} = \frac{1}{208 \times 10^3\,MPa}\left[470.5\,MPa - 0.3(525\,MPa)\right] = 0.00150.
$$

List all assumptions. We have used the thin-walled relation to compute cross-sectional area.
We have assumed that the radial stress can be neglected in calculation of yielding, and that the state of stress is approximately uniform through the wall thickness of the tube.

Note that a slightly more conservative value for the "at yield" pressure value could be obtained by recognizing that, on the inside radius of the tube, the value of the radial stress is $\sigma_{rr} = -p = -(t/\bar{R})\Sigma = -\Sigma/10$. Thus, when the Mises stress expression is evaluated with this (small, but non-zero) value for σ_{rr}, the resulting quadratic equation for Σ is changed; a numerical solution for this case gives $\Sigma = 516\,MPa$ and $p = 51.6\,MPa$, just a couple of percent below our (simpler) estimate, above, that gave $\Sigma = 525\,MPa$ and $p = 52.5\,MPa$.

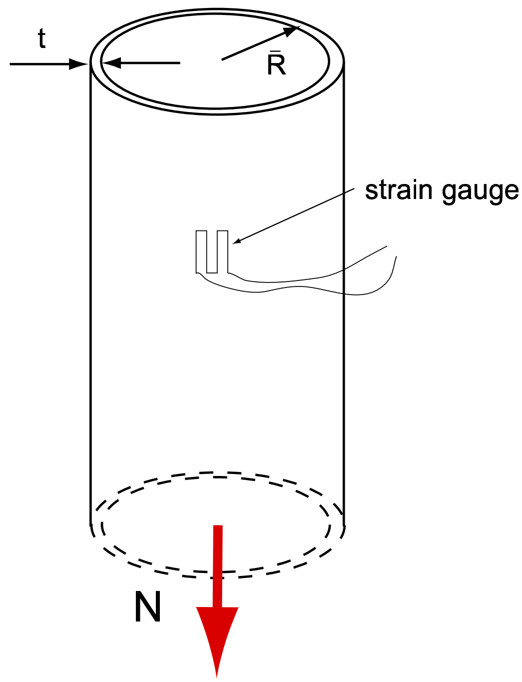

Figure 2: Schematic of thin-walled tube subjected to axial load N and internal pressure, p.

Isotropic Linear Thermal-Elasticity
(Cartesian Coordinates)

Stress/Strain/Temperature-Change Relations:

$$\epsilon_{ij} = \alpha \Delta T \, \delta_{ij} + \frac{1}{E} \left[(1+\nu)\, \sigma_{ij} - \nu \, \left(\sum_{k=1}^{3} \sigma_{kk} \right) \, \delta_{ij} \right].$$

$$\sigma_{ij} = \frac{E}{(1+\nu)} \left[\epsilon_{ij} + \frac{\nu}{(1-2\nu)} \, \left(\sum_{m=1}^{3} \epsilon_{mm} \right) \, \delta_{ij} - \frac{(1+\nu)}{(1-2\nu)} \, \alpha \Delta T \, \delta_{ij} \right].$$

Strain-displacement Relations:

$$\epsilon_{ij} = \frac{1}{2} \, \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right).$$

Equilibrium equations (with body force and acceleration):

$$\sum_{j=1}^{3} \frac{\partial \sigma_{ij}}{\partial x_j} + \rho b_i = \rho \frac{\partial^2 u_i}{\partial t^2}$$

Mises Stress Measure:

$$\bar{\sigma} \equiv \sqrt{\frac{1}{2} \, \left[(\sigma_{11} - \sigma_{22})^2 + (\sigma_{22} - \sigma_{33})^2 + (\sigma_{33} - \sigma_{11})^2 \right] + 3 \, \left[\sigma_{12}^2 + \sigma_{23}^2 + \sigma_{31}^2 \right]}.$$

Traction on elemental area with unit normal "n":

$$t_i(\mathbf{n}) = \sum_{j=1}^{3} \, \sigma_{ij} \, n_j.$$