Polynomial Method

Theorem 0.1 (3D Szemerédi-Trotter)**.** *Given S points and L lines in $\mathbb{R}^3$ with at most B lines in any plane, the number of incidences $I(S,L)$ is at most $S^{\frac{1}{2}}L^{\frac{3}{4}} + B^{\frac{1}{3}}L^{\frac{1}{3}}S^{\frac{2}{3}} + S + L$.*

The four terms of that sum are tight for, respectively, a 3-D grid, L/B planes with B lines in each with the 2-D Szemerédi-Trotter arrangement, all points collinear, and all lines concurrent, respectively.

We already know that $I(S,L) \leq S^2 + L$ and $I(S,L) \leq L^2 + S$ by counting, and $I(S,L) \leq C[L^{\frac{2}{3}}S^{\frac{2}{3}} + L + S]$ by Szemerédi-Trotter. So we're already done unless $S \leq L^2 \leq S^4$ (ignoring constants).

Claim 1 (Cell Estimate). In a polynomial cell decomposition of degree d, $I(S,L) \leq C[d^{-\frac{1}{3}}S^{\frac{2}{3}}L^{\frac{2}{3}} + dL + S_{cell}] + I(S_{alg}, L_{alg})$.

Proof. Let the cells be O_i, and let S_i and L_i be the number of points and lines that intersect O_i. Then $\sum S_i = S_{cell} \leq S$, $\sum L_i \leq dL$, and $S_i \leq d^{-3}S$. (Here and henceforth, we drop constants.) Then $I(S_{cell}, L) = \sum_i I(S_i, L_i) \leq \sum_i S_i^{\frac{2}{3}}L_i^{\frac{2}{3}} + L_i + S_i \leq (d^{-1}S^{\frac{1}{3}}\sum_i S_i^{\frac{1}{3}}L_i^{\frac{2}{3}}) + \sum_i L_i + S_i$. By Hölder's inequality, that's at most $(d^{-1}S^{\frac{1}{3}}(\sum_i S_i)^{\frac{1}{3}}(\sum L_i)^{\frac{2}{3}}) + \sum_i L_i + S_i = d^{-\frac{1}{3}}S^{\frac{2}{3}}L^{\frac{2}{3}} + dL + S_{cell}$.

Finally, $I(S_{alg}, L_{cell}) \leq dL$ by degree bounding, so we've counted everything but $I(S_{alg}, L_{alg})$, as desired. $\square$

Let L_p, L_m and L_u ("planar," "multiplanar," and "uniplanar") be the sets of lines in at least one, at least two, and exactly one plane of $Z(P)$, respectively, and let S_p, S_m, and S_u be the same for points.

Claim 2 (Planar Estimate). $I(S_{alg}, L_p) \leq C[B^{\frac{1}{3}}L^{\frac{1}{3}}S^{\frac{2}{3}} + dL + S_u] + I(S_m, L_m)$.

Also, $|L_m| \leq d^2$; we'll choose d small enough that the last term is handleable by induction.

Proof. $I(S_{alg}, L_p) \leq I(S_{alg}, L_u) + I(S_m, L_m)$, since a line in multiple planes only hits points in multiple planes. Let Π be the set of planes in $Z(P)$.

$I(S_{alg}, L_u) \leq \sum_{\pi\in\Pi} I(S_\pi, L_{u:\pi}) \leq \sum_\pi dL_{u:\pi} + I(S_{u:\pi}, L_{u:\pi})$. By the same application of Hölder's Inequality as before, that's at most $dL + B^{\frac{1}{3}}L^{\frac{1}{3}}S^{\frac{2}{3}} + S_u$. $\square$

That leaves the nonplanar algebraic lines (and multiplanar lines) to bound. We'll use special points, that is, flat or critical points, that is, points at which SP (which has degree at most $3d$) is 0 and special lines, on which every point is special.

Let S_s and S_n be the sets of special and nonspecial points, respectively, in S_{alg}, and define L_s and L_n similarly.

Claim 3 (Algebraic Estimate). $I(S_{alg}, L_{alg} \setminus L_p) \leq C[dL + S_n] + I(S_s, L_s \setminus L_p)$, and $|L_s \setminus L_p| \leq 10d^2$

Proof. Recall that

1. If x is in three lines of $Z(P)$ then x is special,

2. x is special iff $SP(x)$ is 0, where $\deg(SP) \leq 3d$, and

3. The number of lines that are special but not planar is at most $10d^2$.

Now, $I(S_{alg}, L_{alg} \setminus L_p) \leq I(S_n, L_{alg} + I(S_s, L_n) + I(S_s, L_s \setminus L_p)$. The first term is at most $2S_n$ by the first recalled property and the second term is at most $3dL$ by the second recalled property. $\square$

That leaves $I(S_s, L_s \setminus L_p)$ and $I(S_m, L_m)$ to bound; those contain at most $11d^2$ lines. Let $S' = S \setminus (S_s \cup S_m)$. We already have $I(S, L) \leq d^{-\frac{1}{3}} L^{\frac{2}{3}} S^{\frac{2}{3}} + dL + B^{\frac{1}{3}} L^{\frac{1}{3}} S^{\frac{2}{3}} + S' + I(S_s, L_s \setminus L_p) I(S_m, L_m)$.

Lemma 1. The minimum value of $d^{-\frac{1}{3}} L^{\frac{2}{3}} S^{\frac{2}{3}} + dL + B^{\frac{1}{3}} L^{\frac{1}{3}} S^{\frac{2}{3}} + S$ with $d \in [1, \frac{1}{9} L^{\frac{1}{2}}]$ (and $B \geq L^{\frac{1}{2}}$ is about $S^{\frac{1}{2}} L^{\frac{3}{4}} + B^{\frac{1}{3}} L^{\frac{1}{3}} S^{\frac{2}{3}} + S'$

Proof. Just do it. $\square$

So $I(S, L) \leq C[S^{\frac{1}{2}} L^{\frac{3}{4}} + B^{\frac{1}{3}} L^{\frac{1}{3}} S^{\frac{2}{3}} + S'] + C_0[S^{\frac{1}{2}} (\frac{L}{2})^{\frac{3}{4}} + B^{\frac{1}{3}} (\frac{L}{2})^{\frac{1}{3}} S^{\frac{2}{3}} + (S - S')]$, and we can choose C_0 arbitrarily and bigger than, say, $100C$, so that's at most $C_0[S^{\frac{1}{2}} L^{\frac{3}{4}} + B^{\frac{1}{3}} L^{\frac{1}{3}} S^{\frac{2}{3}} + S + L]$, as desired.

0.1 Efficiency of Polynomials

Theorem 0.2 ("Efficiency of Polynomials")**.** *If $P : \mathbb{C} \to \mathbb{C}$ is a polynomial and $F : \mathbb{C} \to \mathbb{C}$ is smooth (not necessarily holomorphic), and $F = P$ outside some bounded domain Ω, and 0 is a regular value of P and F, then P has at most as many zeros in Ω as F does.*

(If $F : M^m \to N^n$ is a function, then $x \in M$ is a critical point iff dF_x isn't surjective, and a regular point otherwise. $y \in N$ is regular iff all its preimages are regular. In our case, if $x \in Z(F)$, that 0 is a regular value implies that $dF_x : \mathbb{R}^2 \to \mathbb{R}^2$ is an isomorphism. Call $\sigma(x)$ 1 if dF_x preserves orientation and -1 otherwise.)

If P is a complex polynomial, then $\sigma_P(x) = +1$ for all $x \in Z(P)$.

Theorem 0.3. *The winding number of $F : \partial\Omega \to \mathbb{C} \setminus \{0\}$ is $\sum_{x \in Z(F) \cap \Omega} \sigma_F(x) = \sum_{x \in Z(P) \cap \Omega} \sigma_P(x)$.*