18.S66 PROBLEMS #6

Spring 2003

Let $\lambda = (\lambda_1, \lambda_2, \ldots) \vdash n$. A *standard Young tableau* (SYT) of shape λ is a left-justified array of the integers $1, 2, \ldots, n$, each occurring exactly once, with λ_i entries in the ith row, such that every row and column is increasing. An example of an SYT of shape $(4, 4, 2)$ is given by

$$\begin{array}{cccc} 1 & 2 & 3 & 6 \\ 4 & 5 & 8 & 10 \\ 7 & 9 & & \end{array}.$$

We write f^λ for the number of SYT of shape λ.

Let u be a square of the Young diagram of λ, denoted $u \in \lambda$. The *hook length* $h(u)$ of u is the number of squares directly to the right or directly below u, counting u itself once. If $u = (i, j)$ (i.e., u is in the ith row and jth column of (the Young diagram of) λ), then $h(u) = \lambda_i + \lambda'_j - i - j + 1$. The hook lengths of $(4, 2, 2)$ are given by

6	5	2	1
3	2		
2	1		

175. [1.5] Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ be a partition, and set $\mu_i = \lambda_i + n - i$. The multisets $\{h(u) : u \in \lambda\} \cup \{\mu_i - \mu_j : 1 \leq i < j \leq n\}$ and $\bigcup_{i=1}^n \{1, 2, \ldots, \mu_i\}$ are equal.

176. [2] Let $\eta_k(\lambda)$ denote the number of hooks of length k of the partition λ. Then
$$\sum_{\lambda \vdash n} \eta_k(\lambda) = k \sum_{\lambda \vdash n} m_k(\lambda).$$
As usual, $m_k(\lambda)$ denotes the number of parts of λ equal to k.

177. [2] Let $\lambda \vdash n$ and $1 \leq i < n$. The number of SYT of shape λ for which $i + 1$ appears in a lower row than i is independent of i.

178. [1] The number of SYT of shape (n,n) is the Catalan number $C_n = \frac{1}{n+1}\binom{2n}{n}$.

179. (*) Let a^λ denote the number of SYT of shape λ with a 2 in the first row. Then

$$a^\lambda = \frac{f^\lambda}{n(n-1)}\left[\binom{n}{2} + \sum \binom{\lambda_i}{2} - \sum \binom{\lambda_i'}{2}\right].$$

180. [1.5] How many SYT of shape $\langle n^n \rangle$ have main diagonal $(1,4,9,16,\ldots,n^2)$?

181. [3] The number of SYT of shape λ is given by

$$f^\lambda = \frac{n!}{\prod_{u\in\lambda} h(u)}.$$

This is the famous *hook-length formula* of Frame, Robinson, and Thrall (1954). It was only given a "satisfactory" bijective proof in 1997.

182. [3] Show that $\sum_{\lambda\vdash n}(f^\lambda)^2 = n!$. In other words, the number of pairs (P,Q) of SYT of the same shape and with n entries is $n!$.

183. [2] With a^λ as in Problem 179, evaluate the sums

$$\sum_{\lambda\vdash n} a^\lambda f^\lambda \quad\text{and}\quad \sum_{\lambda\vdash n} \left(a^\lambda\right)^2.$$

184. [3] The total number of SYT with n entries is equal to the number of involutions $w \in \mathfrak{S}_n$, i.e., $w^2 = 1$.

185. [3] The number of SYT with $2n$ entries and all rows of even length is $1\cdot 3\cdot 5\cdots(2n-1)$.

186. [2] The number of SYT with n entries and at most two rows is $\binom{n}{\lfloor n/2\rfloor}$.

187. [3] The number of SYT with n entries and at most three rows is equal to $\sum_{i=0}^{\lfloor n/2\rfloor}\binom{n}{2i}C_i$, where C_i denotes a Catalan number.

188. [3] The number of SYT with n entries and at most four rows is equal to $C_{\lfloor (n+1)/2\rfloor}C_{\lceil (n+1)/2\rceil}$.

NOTE. There is a similar, though somewhat more complicated, formula for the case of five rows. For six and more rows, no "reasonable" formula is known.

189. [2] The number of pairs (P, Q) of SYT of the same shape with n entries each and at most two rows is the Catalan number C_n.

190. [3] The number of pairs (P, Q) of SYT of the same shape with n entries each and at most three rows is given by

$$\frac{1}{(n+1)^2(n+2)} \sum_{k=0}^{n} \binom{2k}{k} \binom{n+1}{k+1} \binom{n+2}{k+1}.$$

191. [2] Let $W_i(n)$ be the number of ways to draw i diagonals in a convex n-gon such that no two diagonals intersect in their interiors. Then $W_i(n)$ is the number of standard Young tableaux of shape $\langle (i+1)^2, 1^{n-i-3} \rangle$ (i.e., two parts equal to $i+1$ and $n-i-3$ parts equal to 1; when $i = 0$ there are $n-1$ parts equal to 1).

NOTE. Given the result of this problem, it follows immediately from the hook-length formula (Problem 181) that

$$W_i(n) = \frac{1}{n+i} \binom{n+i}{i+1} \binom{n-3}{i},$$

a result originally stated by Kirkman (1857) and Prouhet (1866), with the first complete proof by Cayley (1890-91).

192. [2] Let T be an SYT of shape $\lambda \vdash n$. For each entry of T not in the first column, let $f(i)$ be the number of entries j in the column immediately to the left of i and in a row not above i, for which $j < i$. Define $f(T) = \prod_i f(i)$, where i ranges over all entries of T not in the first column. For instance, if

$$T = \begin{array}{llll} 1 & 3 & 6 & 8 \\ 2 & 4 & 7 & \\ 5 & & & \end{array},$$

then $f(3) = 2$, $f(4) = 1$, $f(6) = 2$, $f(7) = 1$, $f(8) = 2$, and $f(T) = 8$. Then $\sum_{\mathrm{sh}(T)=\lambda} f(T)$, where T ranges over all SYT of shape λ, is equal to the number of partitions of the set $[n]$ of type λ (i.e., with block sizes $\lambda_1, \lambda_2, \ldots$).

193. [3.5] Let $\lambda \vdash n$. An assignment $u \mapsto a_u$ of the distinct integers $1, 2, \ldots, n$ to the squares $u \in \lambda$ is a *balanced tableau* of shape λ if for each $u \in \lambda$ the number a_u is the kth largest number in the hook

of u, where k is the leg-length (number of squares directly below u, counting u itself) of the hook of u. For instance, the balanced tableaux of shape $(3, 2)$ are

$$\begin{matrix} 4\,2\,1 \\ 5\,3 \end{matrix} \quad \begin{matrix} 4\,2\,3 \\ 5\,1 \end{matrix} \quad \begin{matrix} 4\,2\,5 \\ 3\,1 \end{matrix} \quad \begin{matrix} 4\,3\,5 \\ 2\,1 \end{matrix} \quad \begin{matrix} 3\,2\,1 \\ 5\,4 \end{matrix} \; .$$

Let b^λ be the number of balanced tableaux of shape λ. Then $b^\lambda = f^\lambda$, the number of SYT of shape λ.

NOTE. For such a simply stated problem, this seems remarkably difficult to prove.

194. [2] Let $f(n)$ be the number of ways to write the permutation $n, n-1, n-2, \dots, 1 \in \mathfrak{S}_n$ as a product of $\binom{n}{2}$ (the minimum possible) adjacent transpositions $s_i = (i, i+1)$, $1 \le i \le n-1$. For instance, $f(3) = 2$, corresponding to $s_1 s_2 s_1$ and $s_2 s_1 s_2$. Then $f(n)$ is equal to the number of balanced tableaux (as defined in the previous problem) of shape $(n-1, n-2, \dots, 1)$.

NOTE. It thus follows from the previous problem that $f(n) = f^{(n-1,n-2,\dots,1)}$. *Any* bijective proof of this difficult result would be an impressive achievement. (There are known bijective proofs, but they are far from obvious even when the bijection is described.)

195. (*) Let $w_0 = n, n-1, n-2, \dots, 1 \in \mathfrak{S}_n$ and $p = \binom{n}{2}$. Define

$$R_n = \{(a_1, \dots, a_p) \in [n-1]^p \, : \, w_0 = s_{a_1} s_{a_2} \cdots s_{a_p}\},$$

where $s_i = (i, i+1)$ as in the previous problem. For example, $R_3 = \{(1,2,1), (2,1,2)\}$. Then

$$\sum_{(a_1,\dots,a_p)\in R_n} a_1 a_2 \cdots a_p = p!.$$

For instance, when $n = 3$ we get $1 \cdot 2 \cdot 1 + 2 \cdot 1 \cdot 2 = 3!$.

196. [2.5] An *oscillating tableau* of length $2n$ and shape $\emptyset$ is a sequence

$$(\lambda^0, \lambda^1, \dots, \lambda^{2n}),$$

where each λ^i is a partition, $\lambda^0 = \lambda^{2n} = \emptyset$, and each λ^i is obtained from λ^{i-1} by either adding or removing a square from (the diagram of) λ. For instance, when $n=2$ we get the three oscillating tableaux $(\emptyset,1,\emptyset,1,\emptyset)$, $(\emptyset,1,2,1,\emptyset)$, and $(\emptyset,1,11,1,\emptyset)$. The number of oscillating tableaux of length $2n$ and shape $\emptyset$ is equal to $1\cdot 3\cdot 5\cdots(2n-1)$ (the number of partitions of $[2n]$ into n 2-element blocks).

197. [2.5] The number of ways to move from the empty partition $\emptyset$ to $\emptyset$ in n steps, where each step consists of either (i) adding a box, (ii) removing a box, or (iii) adding and then removing a box, always keeping the diagram of a partition (even in the middle of a step of type (iii)), is the Bell number $B(n)$ (the number of partitions of an n-element set). For instance, when $n=3$ we get the five sequences

$$\begin{array}{cccc} \emptyset & (1,\emptyset) & (1,\emptyset) & (1,\emptyset) \\ \emptyset & (1,\emptyset) & 1 & \emptyset \\ \emptyset & 1 & (2,1) & \emptyset \\ \emptyset & 1 & (11,1) & \emptyset \\ \emptyset & 1 & \emptyset & (1,\emptyset) \end{array}.$$

198. (*) Given $\lambda\vdash n$, let H_λ denote the product of the hook lengths of λ, so $H_\lambda = n!/f^\lambda$. Then for $k\in\mathbb{N}$,

$$\sum_{\lambda\vdash n} H_\lambda^{k-2} = \frac{1}{n!}\#\{(w_1,w_2,\dots,w_k)\in\mathfrak{S}_n^k\ :\ w_1^2w_2^2\cdots w_k^2=1\}.$$

199. (*) The *major index* $\mathrm{maj}(T)$ of an SYT T is defined to be the sum of all entries i of T for which $i+1$ appears in a lower row than i. Fix $n\in\mathbb{P}$ and $\lambda\vdash n$, and let $m\in\mathbb{Z}$. Then the number of SYT T of shape λ satisfying $\mathrm{maj}(T)\equiv m \pmod{n}$ depends only on λ and $\gcd(m,n)$.

200. [2.5] Let μ be a partition, and let A_μ be the infinite shape consisting of the quadrant $Q=\{(i,j)\ :\ i<0,\ j>0\}$ with the shape μ removed from the lower right-hand corner. Thus every square of A_μ has a finite hook and hence a hook length. For instance, when $\mu=(3,1)$ we get the diagram

⋮

	10	9	8	6	5	3
	9	8	7	5	4	2
⋯	8	7	6	4	3	1
	6	5	4	2	1	
	3	2	1			

Then the multiset of hook lengths of A_μ is equal to the union of the multiset of hook lengths of Q (explicitly given by $\{1^1, 2^2, 3^3, \ldots\}$) and the multiset of hook lengths of μ.

201. [2.5] Let λ be a partition with distinct parts, denoted $\lambda \models n$. A *shifted standard Young tableau* (SHSYT) of shape λ is defined just like an ordinary standard Young tableau of shape λ, except that each row is indented one space to the right from the row above. An example of an SHSYT of shape $(5, 4, 2)$ is given by

$$\begin{array}{ccccc} 1 & 2 & 3 & 5 & 9 \\ & 4 & 6 & 8 & 11 \\ & & 7 & 10 & \end{array}.$$

Let g^λ denote the number of shifted SYT of shape λ. Prove by a suitable modification of RSK that

$$\sum_{\lambda \models n} 2^{n-\ell(\lambda)} \left(g^\lambda\right)^2 = n!.$$

NOTE. The "shifted analogue" of Problem 184 is the following curious result. Let $\zeta = (1+i)/\sqrt{2} = e^{2\pi i/8}$. Let

$$u(n) = \sum_{\lambda \models n} \zeta^{\ell(\lambda)} 2^{(n-\ell(\lambda))/2} g^\lambda.$$

Then

$$\sum_{n\geq 0} u(n)\frac{t^n}{n!} = e^{\zeta t+\frac{1}{2}t^2}.$$

202. [2.5] A *plane partition* of n is an array $\pi = (\pi_{ij})_{i,j\geq 1}$ of nonnegative integers whose rows and columns are weakly decreasing and whose entries sum to n. When writing π, the entries equal to 0 are often omitted. Thus the plane partitions of the integers $0 \leq n \leq 3$ are given by

$$\begin{array}{ccccccccccc} \emptyset & 1 & 2 & 11 & 1 & 3 & 21 & 111 & 11 & 2 & 1 \\ & & & & 1 & & & & 1 & 1 & 1 \\ & & & & & & & & & & 1. \end{array}$$

If π is a plane partition of n, then we write $|\pi| = n$. Let $a_{rs}(n)$ denote the number of plane partitions of n with at most r rows and at most s columns (of nonzero entries). Then

$$\sum_{n\geq 0} a_{rs}(n)x^n = \prod_{i=1}^{r}\prod_{j=1}^{s}(1-x^{i+j-1})^{-1}. \qquad (7)$$

In particular, let $a(n)$ denote the total number of plane partitions of n. If we let $r, s \to \infty$ in (7) then it's not hard to see that we get

$$\sum_{n\geq 0} a(n)x^n = \prod_{i\geq 1}(1-x^i)^{-i},$$

a famous formula of MacMahon.

HINT. Use the RSK algorithm.

NOTE. At this point it's natural to consider *three-dimensional* (and higher) partitions, but almost nothing is known about them, and a "reasonable" enumeration of them is believed to be hopeless.

203. [3] Fix $r, s, t > 0$. Let $\mathcal{P}(r,s,t)$ denote the set of plane partitions with at most r rows, at most s columns, and with largest part at most t. Then

$$\sum_{\pi\in\mathcal{P}(r,s,t)} x^{|\pi|} = \prod_{i=1}^{r}\prod_{j=1}^{s}\prod_{k=1}^{t}\frac{1-x^{i+j+k-1}}{1-x^{i+j+k-2}}.$$

Note that Problem 202 is the case $t \to \infty$.

204. [3] A plane partition $\pi = (\pi_{ij})$ is *symmetric* if $\pi_{ij} = \pi_{ji}$ for all i, j. Let $b(n)$ denote the number of symmetric plane partitions of n. Then

$$\sum_{n\geq 0} b(n)x^n = \prod_{i\geq 1} \frac{1}{(1-x^{2i-1})\,(1-x^{2i})^{\lfloor i/2 \rfloor}}.$$

205. [2] Let $f_{rs}(n)$ denote the number of plane partitions $\pi = (\pi_{ij})$ with at most r rows, at most s columns, and with *trace* $\mathrm{tr}(\pi) := \pi_{11} + \pi_{22} + \cdots = n$. Then

$$f_{rs}(n) = \binom{rs+n-1}{rs-1}.$$

206. [1.5] A *monotone triangle* of length n is a triangular array of integers whose first row is $1, 2, \ldots, n$, every row is strictly increasing, and each entry is (weakly) between its two neighbors above. This somewhat vague definition should be made clear by the following example:

$$\begin{array}{ccccccccccc} 1 & & 2 & & 3 & & 4 & & 5 & & 6 \\ & 1 & & 2 & & 3 & & 4 & & 6 & \\ & & 1 & & 3 & & 4 & & 5 & & \\ & & & 2 & & 4 & & 5 & & & \\ & & & & 2 & & 5 & & & & \\ & & & & & 3 & & & & & \end{array}.$$

There are for instance seven monotone triangles of length 3, given by

$$\begin{array}{ccccccc} 1\,2\,3 & 1\,2\,3 & 1\,2\,3 & 1\,2\,3 & 1\,2\,3 & 1\,2\,3 & 1\,2\,3 \\ 1\,2 & 1\,2 & 1\,3 & 1\,3 & 1\,3 & 2\,3 & 2\,3 \\ 1 & 2 & 1 & 2 & 3 & 2 & 3 \end{array}.$$

An *alternating sign matrix* is a square matrix with entries $0, \pm 1$, such that the nonzero entries in every row and column alternate $1, -1, 1, -1, \ldots, 1, -1, 1$. (Thus every row and column sum is 1.) An example is

$$\left[\begin{array}{cccccc} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & -1 & 0 & 1 & 0 \\ 1 & -1 & 1 & -1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{array}\right].$$

The number of monotone triangles of length n is equal to the number of $n \times n$ alternating sign matrices.

207. (*) An $n \times n$ *totally symmetric self-complementary (TSSC) plane partition* is a plane partition $\pi = (\pi_{ij})_{i,j=1}^n$ satisfying: (i) π is symmetric (as defined in Problem 204), (ii) every row (and hence every column by symmetry) of π is a self-conjugate partition, and (iii) π is invariant under the operation of replacing each entry i with $n - i$ and rotating 180°. It is easy to see that n must be even. The 4×4 TSSC plane partitions are given by:

$$\begin{bmatrix} 4 & 4 & 2 & 2 \\ 4 & 4 & 2 & 2 \\ 2 & 2 & 0 & 0 \\ 2 & 2 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 4 & 4 & 3 & 2 \\ 4 & 3 & 2 & 1 \\ 3 & 2 & 1 & 0 \\ 2 & 1 & 0 & 0 \end{bmatrix}.$$

A *descending plane partition* is an array of positive integers satisfying: (i) Each row after the first contains fewer elements than the row above, (ii) each row is indented one space to the right from the row above, (iii) the entries weakly decrease in each row, (iv) the entries strictly decrease in each column, (v) the first entry in each row (except the first) does not exceed the number of entries in the preceding row, and (vi) the first entry in each row is greater than the number of entries in its own row. The descending partitions with largest part at most three are given by

$$\emptyset \qquad 2 \qquad 3 \qquad 31 \qquad 32 \qquad 33 \qquad \begin{matrix} 3\,3 \\ \;\;2 \end{matrix}.$$

The following four numbers are all equal:

(a) $\displaystyle\prod_{i=0}^{n-1} \frac{(3i+1)!}{(n+i)!}$

(b) the number of monotone triangles of length n

(c) the number of $2n \times 2n$ TSSC plane partitions

(d) the number of descending plane partitions with largest part at most n

NOTE. There are $\binom{4}{2} = 6$ pairs of equal numbers above. *None* of these six pairs is known to be equal by a bijective proof! (All are known to be equal by complicated indirect arguments.) This is one of the most intriguing open problems in the area of bijective proofs. There are certain refinements of the numbers (a)–(d) which may be useful in finding bijections. For instance, it appears that the number of descending plane partitions with largest part at most n and with exactly k parts equal to n is equal to the number of monotone triangles of length n and bottom element $k+1$. Similarly, it seems that the number of descending plane partitions with largest part at most n and with exactly k parts is equal to the number of monotone triangles of length n with exactly k entries which are greater than the entry to the upper left.

208. [3] If A is an alternating sign matrix, let $s(A)$ denote the number of -1's in A. Then

$$\sum_A 2^{s(A)} = 2^{\binom{n}{2}},$$

where A ranges over all $n \times n$ alternating sign matrices.

209. [3] Let $w = a_1 \cdots a_n \in \mathfrak{S}_n$. An *increasing subsequence* of w of length j is a subsequence $a_{i_1} a_{i_2} \cdots a_{i_j}$ of w (so $i_1 < i_2 < \cdots < i_j$) such that $a_{i_1} < a_{i_2} < \cdots < a_{i_j}$. *Decreasing subsequence* is defined analogously. Let $\mathrm{is}(w)$ (respectively, $\mathrm{ds}(w)$) denote the length of the longest increasing (respectively, decreasing) subsequence of w. A famous result of Erdős and Szekeres, given an equally famous elegant pigeonhole proof by Seidenberg, states that if $n = pq+1$, then either $\mathrm{is}(w) > p$ or $\mathrm{ds}(w) > q$. The number $A(p,q)$ of $w \in \mathfrak{S}_{pq}$ satisfying $\mathrm{is}(w) = p$ and $\mathrm{ds}(w) = q$ is given by $(f^\lambda)^2$, where λ is the partition with p parts equal to q (i.e., the diagram of λ is a $p \times q$ rectangle). Note that the hook-length formula (Problem 181) then gives an explicit formula for $A(p,q)$.

210. [3] If T is an SYT with n entries, then let $w(T)$ be the permutation of $1, 2, \ldots, n$ obtained by reading the entries of T in the usual (English) reading order. For instance, if T is given by

$$\begin{matrix} 1 & 3 & 4 & 9 \\ 2 & 6 & 8 & \\ 5 & 7 & & \end{matrix} \quad ,$$

then $w(T) = 134926857 \in \mathfrak{S}_9$. Define

$$\operatorname{sgn}(T) = \begin{cases} 1, & \text{if } w(T) \text{ is an even permutation} \\ -1, & \text{if } w(T) \text{ is an odd permutation.} \end{cases}$$

Then

$$\sum_T \operatorname{sgn}(T) = 2^{\lfloor n/2 \rfloor}, \tag{8}$$

summed over all SYT with n entries.