Problem Set 3

IAP 2019 18.S097: Applied Category Theory

Good academic practice is expected. In particular, cooperation is encouraged, but assignments must be written up alone, and collaborators and resources consulted must be acknowledged. Please let us know if you consult the Solutions section in the book.

We suggest that you attempt all problems, but we do not expect all problems to be solved.

Question 1. *Matrices of truth values.*

Let $X = \{1, 2, 3\}$, $Y = \{1, 2\}$, and $Z = \{1, 2, 3\}$, and suppose we have profunctors $M \colon X \nrightarrow Y$ given by the matrix

$$\begin{pmatrix} \texttt{false} & \texttt{false} \\ \texttt{false} & \texttt{true} \\ \texttt{true} & \texttt{true} \end{pmatrix}$$

and $N \colon Y \nrightarrow Z$ given by the matrix

$$\begin{pmatrix} \texttt{true} & \texttt{true} & \texttt{false} \\ \texttt{true} & \texttt{false} & \texttt{true} \end{pmatrix}$$

(a) Recall that a profunctor $X \nrightarrow Y$ is given by a functor $X \times Y \to \mathbf{Bool}$, and that a functor $X \times Y \to \mathbf{Bool}$ is a function $X \times Y \to \mathbf{Bool}$ with a special property (ie. it's order-preserving). In particular, such functions correspond to binary relations $R \subseteq X \times Y$. Write down the relations corresponding to profunctors M and N.
(b) Using matrix multiplication over **Bool**, compute the composite functor $M * N$.
(c) Compose the relations you wrote down in (a). Is this the relation corresponding to the composite matrix?

Question 2. *Matrix multiplication, with diagrams.*

Consider the matrices

$$A := \begin{pmatrix} 1 & 2 \\ 0 & 3 \end{pmatrix} \qquad B := \begin{pmatrix} 0 & 0 & 2 \\ 1 & 3 & 1 \end{pmatrix}$$

(a) Draw a signal flow graph corresponding to A.
(b) Draw a signal flow graph corresponding to B.
(c) Connect the signal flow graphs end-to-end.
(d) Compute the matrix product $A * B$.
(e) Does the product matrix represent the composite SFG?

Question 3. *Rings as monoidal abelian groups.*

Prove that monoids in the monoidal category of abelian groups (with tensor product) are rings.

In case you don't understand the question, let's start from the beginning. A *group* is a monoid $(M, *, e)$ such that for all $m \in M$ there exists an $m' \in M$ such that $m * m' = e$ and $m' * m = e$; "a monoid in which every element has an inverse." A group is called *abelian* if the monoid is commutative, $m * m' = m' * m$.

Given two abelian groups A, B, their *tensor product*, denoted $A \otimes B$ is the abelian group whose elements are

$$A \otimes B = \{\text{expressions } (a_1, b_1) + \cdots + (a_n, b_n) \mid n \in \mathbb{N}, a_i \in A, b_i \in B \text{ for all } i\}/\sim$$

where the equivalence relation $\sim$ is generated as follows:

- $(a, 0) \sim 0$ for all $a \in A$,
- $(0, b) \sim 0$ for all $b \in B$,
- $(a, b) + (a', b) \sim (a + a', b)$ for all a, a', b,
- $(a, b) + (a, b') \sim (a, b + b')$ for all a, b, b',
- $0 + x \sim x$ and $x + 0 \sim x$ for all expressions x, and
- if $x \sim y$ and $x' \sim y'$ then $x + x' \sim y + y'$, for all expressions x, y.

A *ring* $(R, +, 0, \times, 1)$ is a set R such that $(R, +, 0)$ is an abelian group, $(R, \times, 1)$ is a monoid, and for all $a, b, c \in R$ we have $a \times (b + c) = (a \times b) + (a \times c)$ and $(a + b) \times c = (a \times c) + (b \times c)$. (We call this last property *distributivity*.)

(a) The category **Ab** of abelian groups has a monoidal structure, where the monoidal product is $\otimes$. What is the monoidal unit I?

(b) Show that a monoid object in $(\mathbf{Ab}, \otimes, I)$, i.e. an abelian group A, a map $*\colon A \otimes A \to A$, and an element $e\colon I \to A$, together satisfying associativity and unitality, is exactly the same as a ring whose underlying abelian group is A.

Question 4. *A monad is a monoid in a category of endofunctors (harder/optional).*

This question assumes a bit of familiarity with composition of natural transformations; the short section on operations with natural transformations on Wikipedia suffices: https://en.wikipedia.org/wiki/Natural_transformation#Operations_with_natural_transformations.

Given a category $\mathcal{C}$, a *monad* (T, μ, η) on $\mathcal{C}$ is a functor $T\colon \mathcal{C} \to \mathcal{C}$, together with natural transformations $\mu\colon T \mathbin{⨟} T \Rightarrow T$ and $\eta\colon \mathrm{id}_{\mathcal{C}} \Rightarrow T$ such that

- $T\mu \mathbin{⨟} \mu = \mu T \mathbin{⨟} \mu$
- $T\eta \mathbin{⨟} \mu = \eta T \mathbin{⨟} \mu = \mathrm{id}_{\mathcal{C}}$.

Monads are ways of encoding algebraic structures like monoids, groups, and rings. They play a fundamental role in both mathematics and computer science. A first example is the free monoid monad, also known as the list monad; whenever one has an adjunction, a monad may be constructed by composing the left adjoint with the right adjoint. Here we see that monads are just monoids in yet another setting.

Given a category $\mathcal{C}$, we can talk about its monoidal category of endofunctors, write $[\mathcal{C}, \mathcal{C}]$. This category has functors $F\colon \mathcal{C} \to \mathcal{C}$ as objects, and natural transformations between them as morphisms. Composition of morphisms is given by composition of natural transformations, and the monoidal product is given by composition of *functors* (!).

(a) Check that $[\mathcal{C}, \mathcal{C}]$, with $\fatsemi$ as monoidal product, has the data of a monoidal category (Rough definition 4.45, without condition (d)). In particular, what object is the monoidal unit, and why is the monoidal product functorial?

(b) Show that a monoid object (Definition 5.65) in $[\mathcal{C}, \mathcal{C}]$ is a monad.

Question 5. *The origins of addition, multiplication, and exponentiation.*

We have emphasised that category theory allows you to see familiar structures as just instances of universal constructions in the right category.

Consider a skeleton of the category of finite sets. In particular, consider the category where the objects are natural numbers, and a morphism $m \to n$ is a function $\{1, 2, \ldots, m\} \to \{1, 2, \ldots, n\}$. Show that addition is coproduct (hint: Example 6.14), multiplication is product (hint: Example 3.87), and exponentiation is adjoint to product (hint: Example 3.72). Also show that 0 is initial (Example 6.4) and 1 is terminal (Example 3.80).

Question 6. *Tell a story.*

You run into a math major friend in the infinite corridor, and they ask about this course. In approximately half a page, explain to them something interesting you learned.

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