18.S34 (FALL, 2007)

GREATEST INTEGER PROBLEMS

NOTE: We use the notation $\lfloor x \rfloor$ for the greatest integer $\leq x$, even if the original source used the older notation $[x]$.

1. (48P) If n is a positive integer, prove that
$$\left\lfloor \sqrt{n} + \sqrt{n+1} \right\rfloor = \left\lfloor \sqrt{4n+2} \right\rfloor.$$

2. (a) Let p denote a prime number, and let m be any positive integer. Show that the exponent of the highest power of p which divides $m!$ is
$$\left\lfloor \frac{m}{p} \right\rfloor + \left\lfloor \frac{m}{p^2} \right\rfloor + \cdots + \left\lfloor \frac{m}{p^s} \right\rfloor,$$
where $p^{s+1} > m$.

   (b) In how many zeros does the number 1000! end, when written in base 10?

3. (a) Prove that the exponent of the highest power of p which divides $\binom{n}{m}$ is equal to the number of carries that occur when n and $m-n$ are added in base p (Kummer's theorem).

   (b) For $n > 1$ a composite integer, prove that not all of
$$\binom{n}{1}, \dots, \binom{n}{n-1}$$
can be divisible by n.

4. Prove that for any positive integers i, j, k,
$$\frac{(3i)!(3j)!(3k)!}{i!j!k!(i+j)!(j+k)!(k+i)!}$$
is an integer.

5. Prove that for any integers $n_1, \dots n_k$, the product
$$\prod_{1 \leq i < j \leq k} \frac{n_j - n_i}{j - i}$$
is an integer.

6. (68IMO) For every natural number n, evaluate the sum

$$\sum_{k=0}^{\infty} \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor = \left\lfloor \frac{n+1}{2} \right\rfloor + \left\lfloor \frac{n+2}{4} \right\rfloor + \cdots + \left\lfloor \frac{n+2^k}{2^{k+1}} \right\rfloor + \cdots.$$

7. A sequence of real numbers is defined by the *nonlinear* first order recurrence

$$u_{n+1} = u_n(u_n^2 - 3).$$

   (a) If $u_0 = 5/2$, give a simple formula for u_n.

   (b) If $u_0 = 4$, how many digits (in base ten) does $\lfloor u_{10} \rfloor$ have?

8. Define a sequence $a_1 < a_2 < \cdots$ of positive integers as follows. Pick $a_1 = 1$. Once $a_1, \ldots, a_n$ have been chosen, let a_{n+1} be the least positive integer not already chosen and not of the form $a_i + i$ for $1 \leq i \leq n$. Thus $a_1 + 1 = 2$ is not allowed, so $a_2 = 3$. Now $a_2 + 2 = 5$ is also not allowed, so $a_3 = 4$. Then $a_3 + 3 = 7$ is not allowed, so $a_4 = 6$, etc. The sequence begins:

$$1, 3, 4, 6, 8, 9, 11, 12, 14, 16, 17, 19, \ldots.$$

   Find a simple formula for a_n. Your formula should enable you, for instance, to compute $a_{1,000,000}$.

9. (a) (Problem A6, 93P; no contestant solved it.) The infinite sequence of 2's and 3's

$$2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 3, 2, 3, 3, 2, 3, 3, 3, 2, \ldots$$

   has the property that, if one forms a second sequence that records the number of 3's between successive 2's, the result is identical to the first sequence. Show that there exists a real number r such that, for any n, the nth term of the sequence is 2 if and only if $n = 1 + \lfloor rm \rfloor$ for some nonnegative integer m.

   (b) (similar in flavor to (a), though not involving the greatest integer function) Let $a_1, a_2, \ldots$ be the sequence

$$1, 2, 2, 3, 3, 4, 4, 4, 5, 5, 5, 6, 6, 6, 6, 7, 7, 7, 7, 8, 8, 8, 8, 9, 9, 9, 9, 9, \ldots$$

of integers a_n defined as follows: $a_1 = 1$, $a_1 \le a_2 \le a_3 \le \cdots$, and a_n is the number of n's appearing in the sequence. Find real numbers $\alpha, c > 0$ such that

$$\lim_{n\to\infty} \frac{a_n}{n^\alpha} = c.$$

10. (Problem B6, 95P; five of the top 204 contestants received at least 9 points (out of 10), and no one received 3–8 points.) For a positive real number α, define

$$S(\alpha) = \{\lfloor n\alpha \rfloor \,:\, n = 1, 2, 3, \ldots\}.$$

Prove that $\{1, 2, 3, \ldots\}$ cannot be expressed as the disjoint union of three sets $S(\alpha)$, $S(\beta)$, and $S(\gamma)$.

11. Let m be a positive integer and k any integer. Define a sequence $a_m, a_{m+1}, \ldots$ as follows:

$$\begin{aligned} a_m &= k \\ a_{n+1} &= \left\lfloor \frac{n+2}{n} a_n \right\rfloor, \quad n \ge m. \end{aligned}$$

Show that there exists a positive integer N and polynomials $P_0(n)$,$P_1(n)$, $\ldots$, $P_{N-1}(n)$ such that for all $0 \le i \le N-1$ and all integers t for which $tN + i \ge m$, we have

$$a_{tN+i} = P_i(t).$$

12. (Problem B1, 97P; 171 of the top 205 contestants received 10 points, and 14 others received 8–9 points.) Let $\{x\}$ denote the distance between the real number x and the nearest integer. For each positive integer n, evaluate

$$F_n = \sum_{m=1}^{6n-1} \min\left(\left\{\frac{m}{6n}\right\}, \left\{\frac{m}{3n}\right\}\right).$$

(Here $\min(a, b)$ denotes the minimum of a and b.)

13. (Problem B4, 98P; 73 of the top 199 contestants received at least 8 points.) Find necessary and sufficient conditions on positive integers m and n so that

$$\sum_{i=0}^{mn-1} (-1)^{\lfloor i/m \rfloor + \lfloor i/n \rfloor} = 0.$$

14. (Problem B3, 01P; 92 of the top 200 contestants received at least 8 points.) For any positive integer n, let $\langle n\rangle$ denote the closest integer to $\sqrt{n}$. Evaluate

$$\sum_{n=1}^{\infty} \frac{2^{\langle n\rangle} + 2^{-\langle n\rangle}}{2^n}.$$

15. (Problem B3, 03P; 152 of the top 201 contestants received at least 8 points.) Show that for each positive integer n,

$$n! = \prod_{i=1}^{n} \operatorname{lcm}\{1, 2, \ldots, \lfloor n/i \rfloor\}.$$

(Here lcm denotes the least common multiple.)

16. Define $a_1 = 1$ and

$$a_{n+1} = \lfloor \sqrt{2a_n(a_n+1)} \rfloor, \quad n \geq 1.$$

Thus $(a_1, \ldots, a_{10}) = (1, 2, 3, 4, 6, 9, 13, 19, 27, 38)$. Show that $a_{2n+1} - a_{2n} = 2^{n-1}$, and find a simple description of $a_{2n+1} - 2a_{2n-1}$.

17. Prove that for all positive integers m, n,

$$\gcd(m, n) = m + n - mn + 2\sum_{k=0}^{m-1} \left\lfloor \frac{kn}{m} \right\rfloor.$$

18. Let a, b, c, d be real numbers such that $\lfloor na \rfloor + \lfloor nb \rfloor = \lfloor nc \rfloor + \lfloor nd \rfloor$ for all positive integers n. Prove that at least one of $a + b, a - c, a - d$ is an integer.

19. Let p be a prime congruent to 1 modulo 4. Prove that

$$\sum_{i=1}^{(p-1)/4} \lfloor \sqrt{ip} \rfloor = \frac{p^2 - 1}{12}.$$

20. Which positive integers can be written in the form $n + \lfloor \sqrt{n} + \frac{1}{2} \rfloor$ for some positive integer n?

21. For n a positive integer, let x_n be the last digit in the decimal representation of $\lfloor 2^{n/2} \rfloor$. Is the sequence $x_1, x_2, \ldots$ periodic?