18.A34 PROBLEMS #7

77. [1] (a) What is the least number of weights necessary to weigh any integral number of pounds from 1 lb. to 63 lb. inclusive, if the weights must be placed on only one of the scale-pans of a balance? Generalize to any number of pounds.

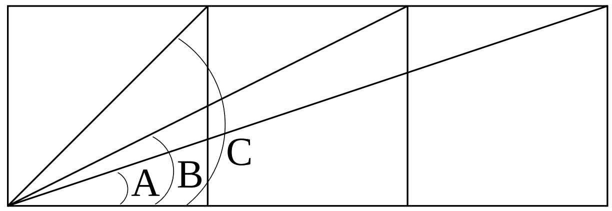
    (b) Same as (a), but from 1 lb. to 40 lb. if the weights can be placed in either of the scale-pans. Generalize.

    (c) A gold chain contains 23 links. What is the least number of links which need to be cut so a jeweler can sell any number of links from 1 to 23, inclusive? Generalize.

78. [2.5] A *perfect partition* of the positive integer n is a finite sequence $a_1 \geq a_2 \geq \cdots \geq a_k$ of positive integers a_i, such that each integer $1 \leq m \leq n$ can be written *uniquely* (regarding equal a_i's as indistinguishable) as a sum of a_i's. For instance, there are three perfect partitions of 5, viz., 11111, 221, and 311, since we have the unique representations 1, $1+1$, $1+1+1$, $1+1+1+1$, $1+1+1+1+1$ in the first case; 1, 2, $2+1$, $2+2$, $2+2+1$ in the second; and 1, $1+1$, 3, $3+1$, $3+1+1$ in the third. Show that the number of perfect partitions of n is equal to the number of *ordered factorizations* of $n+1$ into parts greater than one. For instance, the ordered factorizations of 12 are 12, $6 \cdot 2$, $2 \cdot 6$, $4 \cdot 3$, $3 \cdot 4$, $2 \cdot 2 \cdot 3$, $2 \cdot 3 \cdot 2$, and $3 \cdot 2 \cdot 2$, so there are eight perfect partitions of 11.

79. [1] Here is a proof by induction that all people have the same height. We prove that for any positive integer n, any group of n people all have the same height. This is clearly true for $n = 1$. Now assume it for n, and suppose we have a group of $n+1$ persons, say $P_1, P_2, \ldots, P_{n+1}$. By the induction hypothesis, the n people $P_1, P_2, \ldots, P_n$ all have the same height. Similarly the n people $P_2, P_3, \ldots, P_{n+1}$ all have the same height. Both groups of people contain $P_2, P_3, \ldots, P_n$, so P_1 and P_{n+1} have the same height as $P_2, P_3, \ldots, P_n$. Thus all of $P_1, P_2, \ldots, P_{n+1}$ have the same height. Hence by induction, for any n any group of n people have the same height. Letting n be the total number of people in the world, we conclude that all people have the same height. Is there a flaw in this argument?

80. [1] The following figure consists of three equal squares lined up together, with three diagonals as shown.



Show that angle C is the sum of angles A and B.

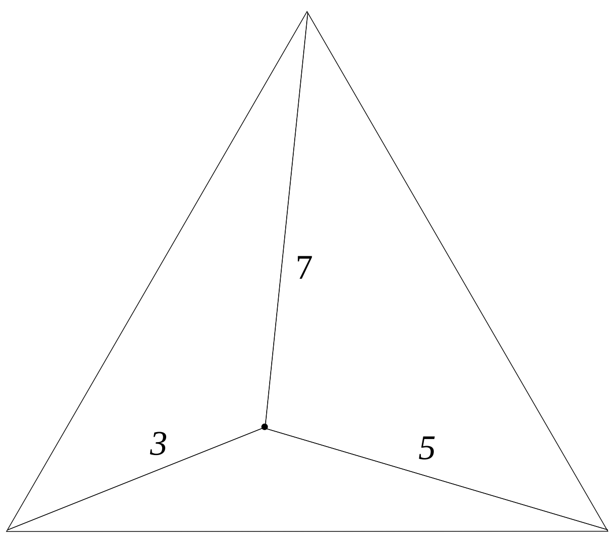
81. [1] Let n be a positive integer.

    (a) Show that if $2^n - 1$ is prime, then n is prime.

    (b) Show that if $2^n + 1$ is prime, then n is a power of two.

    Hint: The simplest way to show that a number is not prime is to factor it explicitly.

82. (a) [2.5] A point P in the interior of an equilateral triangle T is at a distance of 3, 5, and 7 units from the three vertices of T. What is the length of a side of T?



(b) [2.5] More generally, let the point P be at distances a, b, c from the vertices A, B, C of an equilateral triangle of side length d. Find a (nonzero) polynomial equation $f(a, b, c, d) = 0$, symmetric in a, b, c, d.

(c) [2.5] The symmetry of f in a, b, c is obvious, but why also the "hidden symmetry" in d? Find a noncomputational proof.

(d) [2.8] Generalize to n dimensions, i.e., find a (nonzero) polynomial equation $f(a_0, \ldots, a_n, d) = 0$, symmetric in all $n+2$ variables, satisfied by the distances $a_0, \ldots, a_n$ from a point to the vertices of a regular simplex of side length d.

(e) [5] Give a noncomputational explanation of the hidden symmetry of the variable d.

83. [3] Into how few pieces can an equilateral triangle be cut and reassembled to form a square?

84. [2] Let n be an integer greater than one. Show that $1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$ is not an integer.

85. [2.5]

(a) Persons X and Y have nonnegative integers painted on their foreheads which only the other can see. They are told that the sum of the two numbers is either 100 or 101. A third person P asks X if he knows the number on his forehead. If X says "no," then P asks Y. If Y says "no," then P asks X again, etc. Assume both X and Y are perfect logicians. Show that eventually one of them will answer "yes." (This may seem paradoxical. For instance, if X and Y both have 50 then Y knows that X will answer "no" to the first question, since from Y's viewpoint X will see either 50 or 51, and in either case cannot deduce his number. So how does either person gain information?)

(b) Generalize to more than two persons.

86. [2.5] Let $a(n)$ be the exponent of the largest power of 2 dividing the numerator of $\sum_{i=1}^{n} \frac{2^i}{i}$ (when written as a fraction in lowest terms). For instance, $a(1) = 1$, $a(2) = 2$, $a(3) = 2$, $a(4) = 5$. Show that $\lim_{n\to\infty} a(n) = \infty$.

87. [3–] Write the permutation $n, n-1, \ldots, 1$ as a product of $\binom{n}{2}$ (the minimum possible) adjacent transpositions $s_i = (i, i+1)$, $1 \leq i \leq n-1$. For instance, $321 = s_1 s_2 s_1$ (or $s_2 s_1 s_2$). What is the least number of s_i's we need to remove from this product in order to get a product equal to the identity permutation $1, 2, \ldots, n$? For instance, if we remove s_2 from $s_1 s_2 s_1$ then we get $s_1^2 = 123$ (clearly the minimum possible for $n = 3$). Does the answer depend on the way in which we write $n, n-1, \ldots, 1$ as a product of s_i's?