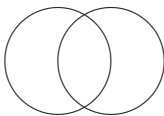


## 18.A34 PROBLEMS #5

51. [1] A person buys a 30-year \$100,000 mortgage at an annual rate of 8%. What is his or her monthly payment?
52. (a) [1] Person  $A$  chooses an integer between 0 and  $2^{11} - 1$ , inclusive. Person  $B$  tries to guess  $A$ 's number by asking yes-no questions. What is the minimum number of questions needed to guarantee that  $B$  finds  $A$ 's number? Can the questions all be chosen in advance in an elegant way?
- (b) [2.5] What if  $A$  is allowed to lie at most once?
53. [1] Let  $M$  be an  $n \times n$  *symmetric* matrix such that each row and column is a permutation of  $1, 2, \dots, n$ . ("Symmetric" means that the entry in row  $i$  and column  $j$  is the same as the entry in row  $j$  and column  $i$ .) If  $n$  is odd, then show that every number  $1, 2, \dots, n$  appears exactly once on the main diagonal. For instance,

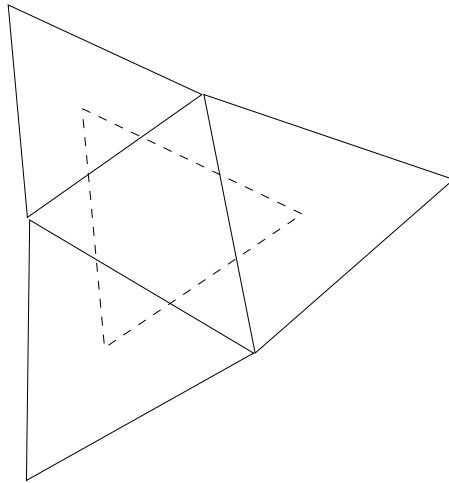
$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 1 & 5 & 3 \\ 3 & 1 & 5 & 2 & 4 \\ 4 & 5 & 2 & 3 & 1 \\ 5 & 3 & 4 & 1 & 2 \end{bmatrix}$$

54. [1] Find all 10 digit numbers  $a_0a_1 \cdots a_9$  such that  $a_i$  is the number of digits equal to  $i$ , for all  $0 \leq i \leq 9$ .
55. [1] Two circles of radius one pass through each other's centers. What is the area of their intersection?



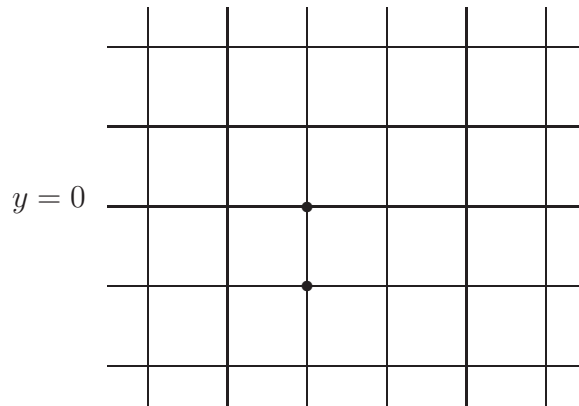
56. (a) [2] Given any 1000 points in the plane, show that there is a circle which contains exactly 500 of the points in its interior, and none on its circumference.

- (b) [3] Given 1001 points in the plane, no three collinear and no four concyclic (i.e., no four on a circle), show that there are exactly 250,000 circles with three of the points on the circumference, 499 points inside, and 499 points outside.
57. (a) [2.5] Let  $n$  be an integer, and suppose that  $n^4 + n^3 + n^2 + n + 1$  is divisible by  $k$ . Show that either  $k$  or  $k - 1$  is divisible by 5. HINT. First show that one may assume that  $k$  is prime. Use *Fermat's theorem* for the prime  $k$ , which states that if  $m$  is not divisible by  $k$ , then  $m^{k-1} - 1$  is divisible by  $k$ . Try to avoid more sophisticated tools.
- (b) [2] Deduce that there are infinitely many primes of the form  $5j + 1$ .
58. [2] A cylindrical hole is drilled straight through and all the way through the center of a sphere. After the hole is drilled, its length is six inches. What is the volume that remains?
59. [2.5] Let  $T$  be a triangle. Erect an equilateral triangle on each side of  $T$  (facing outwards). Show that the centers of these equilateral triangles form the vertices of an equilateral triangle.

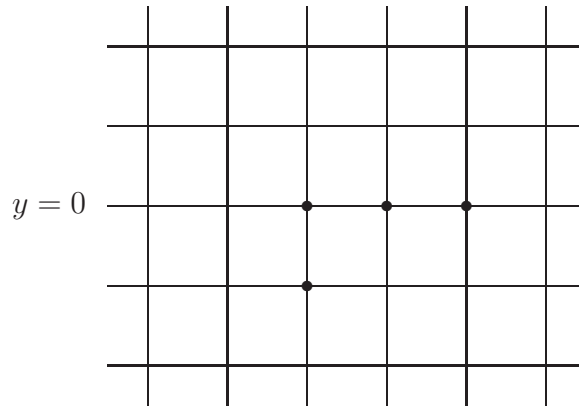


60. [5] Define a sequence  $X_0, X_1, \dots$  of rational numbers by  $X_0 = 2$  and  $X_{n+1} = X_n - \frac{1}{X_n}$  for  $n \geq 0$ . Is the sequence bounded?

61. Let  $B = \mathbb{Z} \times \mathbb{Z}$ , regarded as an infinite chessboard. (Here  $\mathbb{Z}$  denotes the set of integers.) Suppose that counters are placed on some subset of the points of  $B$ . A counter can jump over another counter one step vertically or horizontally to an empty point, and then remove the counter that was jumped over. Given  $n > 0$ , let  $f(n)$  denote the least number of counters that can be placed on  $B$  such that all their  $y$ -coordinates are  $\leq 0$ , and such that by some sequence of jumps it is possible for a counter to reach a point with  $y$ -coordinate equal to  $n$ . For instance,  $f(1) = 2$ , as shown by the following diagram.



Similarly  $f(2) = 4$ , as shown by:



- (a) [2] Show that  $f(3) = 8$  (or at least that  $f(3) \leq 8$  by constructing a suitable example).
- (b) [2.5] Show that  $f(4) = 20$  (or at least that  $f(4) \leq 20$ ).

- (c) [3] Find an upper bound for  $f(5)$ .
62. [3.5] Generalize Problem 12 to  $n$  dimensions as follows. Show that there exist  $n + 1$  lattice points (i.e., points with integer coordinates) in  $\mathbb{R}^n$  such that any two of them are the same distance apart if and only if  $n$  satisfies the following conditions:
- (a) If  $n$  is even, then  $n + 1$  is a square.
  - (b) If  $n \equiv 3 \pmod{4}$ , then it is always possible.
  - (c) If  $n \equiv 1 \pmod{4}$ , then  $n + 1$  is a sum of two squares (of nonnegative integers). The well-known condition for this is that if  $n + 1 = p_1^{a_1} \cdots p_r^{a_r}$  is the factorization of  $n + 1$  into prime powers, then  $a_i$  is even whenever  $p_i \equiv 3 \pmod{4}$ .
63. [5+] Let  $H_n = \sum_{j=1}^n 1/j$ . Show that for all  $n \geq 1$ ,

$$\sum_{d|n} d \leq H_n + (\log H_n)e^{H_n}.$$

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