18.A34 PROBLEMS #1

Problems are marked by the following difficulty ratings.

[1] Easy. Most students should be able to solve it.

[2] Somewhat difficult or tricky. Many students should be able to solve it.

[3] Difficult. Only a few students should be able to solve it.

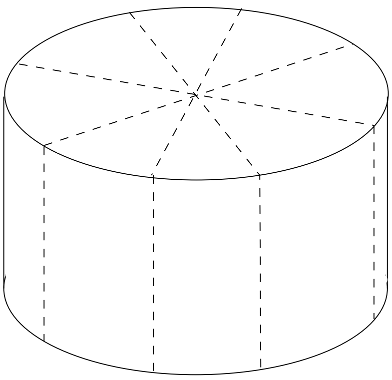
[4] Horrendously difficult. We don't really expect anyone to solve it, but those who like a challenge might want to give it a try.

[5] Unsolved.

Further gradations are indicated by $+$ and $-$. Thus [1–] denotes an utterly trivial problem, and [5–] denotes an unsolved problem that has received little attention and may not be too difficult. A few students may be capable of solving a [3–] problem, while almost none could solve a [3] in a reasonable period of time. Of course these ratings are subjective, so you shouldn't take them *too* seriously.

1. [1] A single elimination tennis tournament is held among 215 players. A player is eliminated as soon as (s)he loses a match. Thus on the first round there are 107 matches and one player receives a bye (waits until the next round). On the second round there are 54 matches with no byes, etc. How many matches are played in all? What if there are 10^{100} players?

2. (a) [1] It's easy to see that a cylinder of cheese can be cut into eight identical pieces with four straight cuts.



Can this be done with only three straight cuts?

(b) [2+] What about a torus (doughnut)? What is the most number of pieces into which a solid torus can be cut by three straight cuts, or more generally by n straight cuts (without rearranging the pieces)?

(c) [3–] What is the most number of pieces into which a solid torus can be cut by three straight cuts, if one is allowed to rearrange the pieces after each cut? So far as I know, the answer is not known for $n \geq 4$ cuts.

3. [2] Can a cube of cheese three inches on a side be cut into 27 one-inch cubes with five straight cuts? What if one can move the pieces prior to cutting?

4. (a) [1] In how many zeros does 10000! end?

   (b) [3] What is the last nonzero digit of 10000!? (No fair using a computer to actually calculate 10000!.)

5. [1] In this problem, "knights" always tell the truth and "knaves" always lie. In (a)-(c), all persons are either knights or knaves.

   (a) There are two persons, A and B. A says, "At least one of us is a knave." What are A and B?

   (b) A says, "Either I am a knave or B is a knight." What are A and B?

   (c) Now we have three persons, A, B, and C. A says, "All of us are knaves." B says, "Exactly one of us is a knight." What are A, B, and C?

   (d) Now we have a third type of person, called "normal," who sometimes lies and sometimes tells the truth. A says, "I am normal." B says, "That is true." C says, "I am not normal." Exactly one of A, B, C is a knight, one is a knave, and one is normal. What are A, B, and C?

6. [1] Without using calculus, find the minimum value of $x + \frac{1}{x}$ for $x > 0$. What about $x + \frac{3}{x}$?

7. [3] Let n and k be nonnegative integers, with $n \geq k$. The *binomial coefficient* $\binom{n}{k}$ is defined by $\binom{n}{k} = n!/k!(n-k)!$. (Recall that $0! = 1$. If $n < k$, then it is convenient to define $\binom{n}{k} = 0$.)

(a) For what values of n and k is $\binom{n}{k}$ odd? Find as simple and elegant a criterion as possible.

(b) More generally, given a prime p, find a simple and elegant description of the largest power of p dividing $\binom{n}{k}$.

8. [3] (a) Let P be a convex polygon in the plane with a prime number p of sides, all angles equal, and all sides of rational length. Show that P is regular (i.e., all sides also have equal length).

   (b) (1990 Olympiad) Show that there exists an equiangular polygon with side lengths $1^2, 2^2, \ldots, 1990^2$ (in some order).

9. [2] What positive integers can be expressed as the sum of two or more consecutive positive integers? (The first three are $3 = 1 + 2$, $5 = 2 + 3$, $6 = 1 + 2 + 3$.)

10. [2] (a) Find the maximum value of $x^{1/x}$ for $x > 0$.

    (b) Without doing any numerical calculations, decide which is bigger, π^e or e^π.

11. [3] Does there exist an infinite sequence $a_0 a_1 a_2 \cdots$ of 1's, 2's, and 3's, such that no two consecutive blocks are identical? (In other words, for no $1 \leq i < j$ do we have $a_i = a_j, a_{i+1} = a_{j+1}, \ldots, a_{j-1} = a_{2j-i-1}$.) For instance, if we begin our sequence with 1213121, then we're stuck.

12. [2] A *lattice point* in the plane is a point with integer coordinates. For instance, $(3, 5)$ and $(0, -2)$ are lattice points, but $(3/2, 1)$ and $(\sqrt{2}, \sqrt{2})$ are not. Show that no three lattice points in the plane can be the vertices of an equilateral triangle. What about in three dimensions?

13. [3.5] True or false? Let n be a positive integer. Then

$$\left\lceil \frac{2}{2^{1/n} - 1} \right\rceil = \left\lfloor \frac{2n}{\log 2} \right\rfloor .$$