**18.997 Topics in Combinatorial Optimization** February 3rd, 2004

Lecture 1

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# 1 Nonbipartite Matching

Our first topic of study is matchings in graphs which are not necessarily bipartite. We begin with some relevant terminology and definitions. A *matching* is a set of edges that share no endvertices. A vertex $v$ is *covered* by a matching if $v$ is incident with an edge in the matching. A matching that covers every vertex is known as a *perfect matching* or a 1-*factor* (i.e., a spanning regular subgraph in which every vertex has degree 1). We will let $\nu(G)$ denote the cardinality of a maximum matching in graph $G$. A *vertex cover* is a set $C$ of vertices such that every edge is incident with at least one vertex in $C$. The minimum cardinality of a vertex cover is denoted $\tau(G)$. The following simple proposition relates matchings and vertex covers.

**Proposition 1** *If $M$ is a matching and $C$ is a vertex cover then $|M| \leq |C|$.*

**Proof:** For each edge in $M$, at least one of the endvertices must be in $C$, since $C$ covers every edge. Since the edges in $M$ do not share any endvertices, we must have $|M| \leq |C|$. □

This proposition implies that $\nu(G) = \max_M |M| \leq \min_C |C| = \tau(G)$, so $\nu(G) \leq \tau(G)$. Kőnig showed that in fact equality holds if $G$ is a bipartite graph with no isolated vertices. Unfortunately if $G$ is not bipartite then we may have $\nu(G) < \tau(G)$. For example, if $G$ is the cycle on three vertices then $\nu(G) = 1$ but $\tau(G) = 2$. We will give another upper-bound for $\nu(G)$ after introducing some more definitions.

If $G = (V, E)$ is a graph and $U \subseteq V$, $G - U$ denotes the subgraph of $G$ obtained by deleting the vertices of $U$ and all edges incident with them. Let $o(G - U)$ denote the number of components of $G - U$ that contain an odd number of vertices. Let $M$ be a matching in $G - U$ and consider a component of $G - U$ with an odd number of vertices. There must be at least one unmatched vertex $v$ in this component, since any matching necessarily covers an even number of vertices. Treating $M$ as a matching in $G$, it is possible that we could increase the size of $M$ by matching $v$ with some vertex in $U$. However, we can add at most $|U|$ edges to $M$ in this manner, since the vertices in $U$ will eventually all be matched. Thus any matching in $G$ must have least $o(G - U) - |U|$ unmatched vertices. This argument shows that the maximum size of a matching is upper-bounded by $(|V| + |U| - o(G - U))/2$, for any subset $U$. The following theorem strengthens this result.

**Theorem 2 (Tutte-Berge Formula)** *Let $G = (V, E)$ be a graph. Then*

$$\nu(G) = \max_M |M| = \min_{U \subset V} (|V| + |U| - o(G - U))/2,$$

*where the maximization is over all matchings $M$ in $G$.*

**Proof:** We will consider the case that $G$ is connected. If $G$ is not connected, the result follows by adding the formulas for the individual components. The proof proceeds by induction on the order of $G$. If $G$ has at most one vertex then the result holds trivially. Otherwise, suppose that $G$ has at least two vertices. We consider two cases.

*Case 1:* $G$ contains a vertex $v$ that is covered by *all* maximum matchings. The subgraph $G - v$ cannot have a matching of size $\nu(G)$, otherwise that would give a maximum matching for $G$ that leaves $v$ unmatched. Thus $\nu(G - v) = \nu(G) - 1$. By induction the result holds for the graph

$G - v$, so there exists a set $U' \subset V - v$ that achieves equality in the Tutte-Berge Formula. Defining $U = U' \cup \{v\}$, we see that

$$
\begin{aligned}
\nu(G) &= \nu(G - v) + 1 \\
&= (|V - v| + |U'| - o(G - v - U'))/2 + 1 \\
&= ((|V| - 1) + (|U| - 1) - o(G - U))/2 + 1 \\
&= (|V| + |U| - o(G - U))/2
\end{aligned}
$$

*Case 2:* For every vertex $v \in G$, there is a maximum matching that does not cover $v$. We will prove that each maximum matching leaves exactly one vertex uncovered. Suppose to the contrary, that is, each maximum matching leaves at least two vertices uncovered. We choose a maximum matching $M$ and its two uncovered vertices $u$ and $v$ such that we minimize $d(u, v)$, the distance between vertices $u$ and $v$. If $d(u, v) = 1$ then the edge $uv$ can be added to $M$ to obtain a larger matching, which is a contradiction.

Otherwise, $d(u, v) \geq 2$ so we may fix an intermediate vertex $t$ on some shortest $u$-$v$ path. By the assumption of the present case, there is a maximum matching $N$ that does not cover $t$. Furthermore, we may choose $N$ such that its symmetric difference with $M$ is minimal. If $N$ does not cover $u$ then $(N, u, t)$ contradicts our choice of $(M, u, v)$. Thus $N$ covers $u$ and, by symmetry, $v$ as well. Since $N$ and $M$ both leave at least two vertices uncovered, there exists a second vertex $x \neq t$ that is covered by $M$ but not by $N$. Let $xy$ be the edge in $M$ that is incident with $x$. If $y$ is also uncovered by $N$ then $N + xy$ is a larger matching than $N$, a contradiction. So let $yz$ be the edge in $N$ that is incident with $y$, and note that $z \neq x$. Then $N + xy - yz$ is a maximum matching that does not cover $t$ and has smaller symmetric difference with $M$ than $N$ does. This contradicts our choice of $N$, so each maximum matching must leave exactly one vertex uncovered. Then $\nu(G) = (|V| - 1)/2$. The Tutte-Berge Formula then follows by choosing $U = \emptyset$. □

A natural question to ask next is: Given a graph $G$, what is a set $U \subset V(G)$ giving equality in the Tutte-Berge Formula? Such a set is provided by the **Edmonds-Gallai Decomposition** of $G$. This decomposition partitions $V(G)$ into three sets: $D(G)$ is the set of all vertices $v$ such that there is some maximum matching that leaves $v$ uncovered, $A(G)$ is the neighbour set of $D(G)$, and $C(G)$ is the set of all remaining vertices.

**Theorem 3** *The set $U = A(G)$ gives equality in the Tutte-Berge Formula. The set $D(G)$ contains all vertices in odd components of $G - U$, and $C(G)$ contains all vertices in even components of $G - U$.*

Let $G[\,D(G)\,]$ be the subgraph of $G$ induced by $D(G)$. It turns out that every connected component $H$ of $G[\,D(G)\,]$ is *factor critical*, meaning that $H - v$ has a perfect matching for every $v \in V(H)$. Thus for any odd component in $G[\,D(G)\,]$ we can actually choose any particular vertex to be left uncovered.

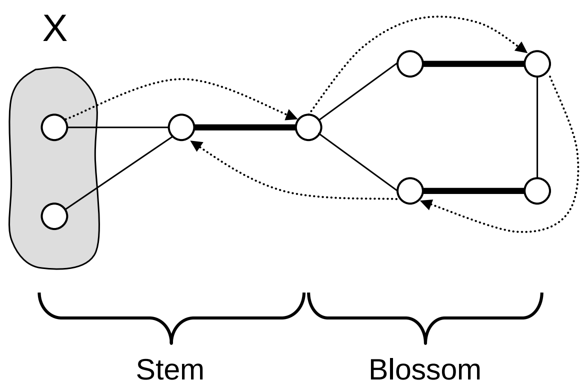
The Edmonds-Gallai Decomposition of a graph can be found as a byproduct of Edmonds' algorithm for finding a maximum matching. Before describing this algorithm, we need some more basic results. Let $M$ be a matching in a graph $G$. An *alternating path* (relative to $M$) is a path $P$ whose edges are alternately in $M$ and not in $M$. An *augmenting path* for $M$ is an alternating path with both endvertices uncovered by $M$. Let $M'$ be the matching obtained by switching $M$-edges and non-$M$-edges along path $P$ (i.e., $M' = M \triangle E(P)$). Then $|M'| = |M| + 1$, which explains why $P$ is called an augmenting path.

**Theorem 4 (Berge)** *$M$ is a maximum matching if and only if $G$ contains no $M$-augmenting path.*

**Proof:** The "only if" direction is trivial, since any augmenting path can be used to increase the size of $M$. To prove the other direction, suppose that $M$ is not maximum and let $N$ be a maximum matching chosen with minimum symmetric difference with $M$. Consider the subgraph spanned by

$M \cup N$. Each vertex has degree at most 2, so the subgraph is a disjoint union of paths and cycles. There are no cycles or paths with equal number of edges from $N$ and $M$, since $N \triangle M$ is minimum. There are no paths with more $N$-edges than $M$-edges otherwise $N$ would not be maximum. It follows that every component is an augmenting path for $M$. □

Theorem 4 implies the following approach for finding a maximum matching: start with an empty matching and repeatedly find augmenting paths to increase its size. **Edmonds' Algorithm** uses this approach and gives a specific method for finding augmenting paths. Consider a graph $G = (V, E)$ and a matching $M$ in $G$. Let $X$ be the set of uncovered vertices in $G$. To find an augmenting path for $M$, it will be helpful to define an auxiliary directed graph $G'$ with vertex set $V$ and arc set $A = \{uv \mid \exists x \in V \text{ such that } ux \in E \text{ and } xv \in M\}$. Observe that a directed path in $G'$ corresponds to an (even length) alternating path in $G$. Furthermore, if there is an augmenting path for $M$ then there is a directed path in $G'$ starting at a vertex in $X$ and ending at a neighbour of $X$. Unfortunately, the converse does not necessarily hold: $G$ may contain a directed path in $G'$ starting at a vertex in $X$ and ending at a neighbour of $X$ that does *not* correspond to an augmenting path. Such a path must necessarily have a prefix that is a *flower*, as shown in this figure.




The dotted arcs show a directed path in the auxiliary graph that starts at a vertex in $X$ and ends at a neighbour of set $X$ but does not correspond to an augmenting path. The graph contains flower, which consists of a *stem* and a *blossom*. The stem is simply an alternating path and the blossom is an odd-length cycle.