4/14. T is a first order stable theory. Assume it has Q.E. in $\mathcal{L}$.

Define $\mathcal{L}_\sigma = \mathcal{L} \cup \{\sigma\}$ σ is a unary function symbol.

Let $T_\sigma = T \cup \{$"σ is an automorphism"$\}$

$$= T \cup \{\forall x\, \varphi(x) \Leftrightarrow \varphi(\sigma(x))\}.$$

Open Problem: Does T_σ have a model companion?

(T, T' are companions if every model of one embeds in a model of the other ($\Leftrightarrow T_\forall = T'_\forall$).
T' is a model companion of T if they are companions, and in addition T' is model complete.)

(e.g. if T has QE, T is model complete.
$T = Th(\mathbb{R})$ "
T is model complete $\Leftrightarrow$ every formula is equiv to an existential formula.

If T is a universal theory, then its model companion is the theory of the class of existentially closed models of T, provided its an elementary class.

If T (still universal) does not have a model companion, one can still define
$\Delta = \{$existential formulas$\}$
$\rightsquigarrow$ T is positive Robinson wrt Δ $\rightsquigarrow$ (so we can construct) universal domain for class of existentially closed models.

Assume: T_σ does have a model companion T_A.

We know that every formula $\varphi(x) \in \mathcal{L}_\sigma$,

$$T_A \vdash \forall x\,[\varphi(x) \leftrightarrow \exists y\, \psi_\varphi(x,y)] \quad \text{where } \psi_\varphi \text{ is q.f.}$$

If $(M,\sigma) \models T_A$ is sufficiently saturated, every possible extension of σ (on something small wrt saturation) is already realised in M.

PAPA^{ann} = "Propriété d'amalgamation de paires d'automorphismes"

"a concept, a way of life :)"

PAPA over models: (can do same for alg closed^{eq} sets)

Assume $M, N_1, N_2 \models T$ s.t. $M \preceq N_1$, $M \preceq N_2$

Assume moreover that $\sigma \in \mathrm{aut}(M)$
$\sigma_1 \in \mathrm{Aut}(N_1)$ extending σ
$\sigma_2 \in \mathrm{Aut}(N_2)$ extending σ

(ie $(M,\sigma), (N_i,\sigma_i) \models T_\sigma$ $(M,\sigma) \leq (N_i,\sigma_i)$ ↑ follows by QE).

Then $\exists\ (P,\bar\sigma) \models T_\sigma$ and embeddings

$$\begin{tikzcd} & (N_1,\sigma_1) \arrow[dr] \arrow[d, phantom, "\circlearrowright"] & \\ (M,\sigma) \arrow[ur] \arrow[dr] & & (P,\bar\sigma) \\ & (N_2,\sigma_2) \arrow[ur] & \end{tikzcd}$$

(Stable theories have PAPAs over models).

Theorem T stable $\Rightarrow$ T has PAPA over models.

Proof: Embed M, N_1, N_2 in a $\cancel{(|N_1|+|N_2|)}^{+}$ very (strongly) homogeneous model $P \models T$ s.t. $M \subseteq N_i$ and

$$N_1 \underset{M}{\downarrow} N_2$$

Let $\bar{a} \in N_1$, $\bar{b} \in N_2$. Then claim $\sigma_1(\bar{a})\ \sigma_2(\bar{b}) \equiv \bar{a}\bar{b}$.

Why? $tp(\bar{a}/M)$ is strong. Since $\bar{a} \underset{M}{\downarrow} \bar{b}$,

$tp(\bar{a}/M)$ determines $tp(\bar{a}/M\bar{b})$ by stationarity.

Let σ' extend σ_2 to an automorphism of P.

$$\text{So} \quad \sigma_1(\bar{a})\ \sigma_2(\bar{b}) \equiv \sigma'^{-1}(\sigma_1(\bar{a})\ \sigma_2(\bar{b})) = \sigma'^{-1} \circ \sigma_1(\bar{a}), \bar{b}$$

$$\bar{a}\bar{b} \equiv \sigma'(\bar{a}\bar{b}) = \sigma'(\bar{a})\ \sigma_2(\bar{b}).$$

All that's left to show is: $\sigma_1(\bar{a}) \equiv_{\sigma_2(\bar{b})} \sigma'(\bar{a})$.

So we have $\bar{a} \underset{M}{\downarrow} \bar{b} \Rightarrow \sigma'\bar{a} \underset{\sigma' M}{\downarrow} \sigma'\bar{b} = \sigma_2\bar{b}$. ①

$\sigma' M = \sigma_2 M = \sigma M$.

Also know $\sigma_1\bar{a} \underset{M}{\downarrow} \sigma_2\bar{b}$ since $N_1 \underset{M}{\downarrow} N_2$. ②

$M = \sigma M$

Moreover $\sigma_1\bar{a}\ \sigma M \equiv \bar{a} M \equiv \sigma'\bar{a}\ \sigma' M$

$\sigma M = \sigma_1 M$, $\sigma' M = \sigma M$

$\Rightarrow \sigma_1 \bar{a} \equiv_{\sigma M} \sigma' \bar{a}$.

Since this is a strong type, $\sigma_1 \bar{a} \equiv_{\sigma M \sigma_2 \bar{b}} \sigma' \bar{a}$.

So we now have $\sigma_1 \bar{a}\, \sigma_2 \bar{b} \equiv \bar{a}\bar{b}$.

Conclusion: $\sigma_1 \cup \sigma_2$ is a partial aut. of P and so extends to an Aut $\bar{\sigma}$. □

<u>Theorem'</u>: T ~~has~~ stable has the PAPA over acl^{eq}-closed sets.

<u>Proof</u>: same.

<u>Theorem''</u>: T a stable CAT has PAPA over $|T|^+$-saturated models.

Proof: same.

~~Lemma~~

<u>Defn</u> Let $\Phi = \{\varphi(x,y) \in \mathcal{L} \mid \varphi \vdash y$ is algebraic over $x\}$ ie $\exists n \quad T \vdash \forall x\, \exists_{\leq n} y\, \varphi(x,y)$

$$\Phi_\sigma = \{\varphi(\sigma^{n_0}(x_0), \sigma^{n_1}(x_1), \ldots, \sigma^{m_0}(y_0), \sigma^{m_1}(y_1), \ldots) : \varphi(\bar{x}, \bar{y}) \in \Phi\}$$

Assume $(M,N) \models T_A$, $\bar{a} \in M$, $\bar{b} \in N$ and also that $\forall \varphi(x,y) \in \Phi_\sigma$ if $M \models \exists y\, \varphi(\bar{a},y)$ then $N \models \exists y\, \varphi(\bar{b},y)$.

Then there exists an $\mathcal{L}_\sigma$ isomorphism $f : \underbrace{acl^{eq}(\sigma^{\mathbb{Z}}(\bar{a}))}_{\text{call } acl^{eq}_\sigma(\bar{a})}$ (σ closure of)

$acl^{eq}_\sigma(\bar{b})$.

~~pre~~ and commutes with σ.

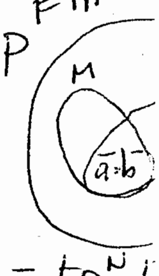
<u>Proof</u> exercise Take $\mathcal{L}_\sigma$-diagram & embed & sce

<u>Corollary</u> Under the assumptions, $\bar{a} \equiv \bar{b}$.

<u>Proof</u>

$$\begin{array}{ccc} M & & N \end{array}$$

$acl^{eq}_\sigma \bar{a} \xrightarrow{\sim} acl_\sigma \bar{b}$ embed into $P \models T_A$ by PAPA



$\hookrightarrow \Rightarrow M, N \preccurlyeq P$

$\Rightarrow tp^M \bar{a} = tp^P \bar{a} = tp^P \bar{b} = tp^N \bar{b}$

In particular: if $\forall \varphi(x,y) \in \Phi_\sigma$, we have $\models \exists y\, \varphi(\bar{a}, \ldots$ $\Leftrightarrow \models \exists y\, \varphi(\bar{b}, \ldots$

then $\bar{a} \equiv \bar{b}$. It follows: every formula equivalent to a boolean combination of for of the form $\exists y\, \varphi(x,y)$, $\varphi \in \Phi_\sigma$.

Exercise: How to express $\forall y\, \neg\varphi(x,y)$ as $\exists z\, \psi(x,z)$ wh $\psi \in \Phi_\sigma$ as well. Remember $\exists n\, \varphi \vdash y$ has at most n conjugates / x.

Lemma 1 Bounded typedefinable sets of hyperimaginaries have hyperimaginary "codes" (canonical parameters).

Namely if $p(x)$ is a partial type with parameters a, " $p(x,a)$

and ~~$p(x,a)$ has~~ $B := \{b : \models p(x,a)\}$ is bounded, then there exists c s.t. an automorphism fixes c iff it fixes B setwise.

Proof Let $\bar{b} = \{b_i : i < \lambda\}$ be an enumeration of B.

Let $r(\bar{x}, y) = tp(\bar{b}, a)$. Let $E(y, y') := [\exists \bar{x}\, r(\bar{x}, y) \wedge r(\bar{x}, y')] \vee y = y'$

Then E is a type-definable equivalence relation.

~~Also: $a E a' \Leftrightarrow B =$~~

Enumerate all formulas $\varphi(x, x') \vdash x \neq x'$ (i.e. $T \vdash \forall x \neg\varphi(x,x)$). Enumerate them as $\{\varphi_i(x, x') : i < \lambda\}$.

For every $i < \lambda$ $\exists n_i < \omega$ s.t.:

1. $\exists x_j$ for $j < n_i$ s.t. $\bigwedge_{j<n_i} p(x_j, a) \wedge \bigwedge_{j<k<n_i} \varphi_i(x_j, x_k)$
2. $\neg$ " $n_i + 1$ st. " $n_i + 1$ " $n_i + 1$ "

(Since with ω this is inconsistent, so let n_i be maximal such that it is.)

$$E(y, y') = (y = y') \vee \Big(\bigwedge_{i<\lambda} \exists x_0 \cdots x_{n_i} \bigwedge_{j<n_i} p(x_j, y) \wedge p(x_j, y') \wedge \bigwedge_{j<k<n_i} \varphi_i(x_j, x_k) \Big) \wedge y, y' \models tp(a) \Big)$$

Clearly: if $a' \models tp(a)$ and $B = \{b : p(b, a')\}$ then $a E a'$.

Now prove converse.

Conversely, assume $a E a'$. So $a' \models tp(a)$.

* IOU rest of proof later...

<u>Lemma 2</u> Every hyperimaginary is interdefinable with a tuple of "small" hyperimaginaries, where small means quotient of a tuple of length $\leq |T|$.

(Proved in an earlier lecture.)

Let T be stable (not necessarily f.o.), M is a $|T|^+$-saturated model & $\sigma \in Aut(M)$.

Assume $A, B, C \supseteq M$, independent over M and boundedly-closed. (ie $A \underset{M}{\perp} BC$ etc)

Moreover, we have $\sigma_A \in Aut(A)$ extending σ,
$\sigma_B \in Aut(B)$ ext σ,
$\sigma_C \in Aut(C)$ ext σ.

Finally, we have $\sigma_{AB} \in Aut(bdd(AB))$ extending $\sigma_A \cup \sigma_B$
σ_{BC} - - -
σ_{AC} - - - -

<u>THEN</u> $\sigma_{AB} \cup \sigma_{AC} \cup \sigma_{BC}$ is elementary. (ie preserves the logic)

<u>Proof</u> Each of $\underbrace{\sigma_{AB} \cup \sigma_{BC}}$ & $\sigma_{AB} \cup_{AC}$ & $\sigma_{BC} \cup \sigma_{AC}$ is elementary.

Since B is bdd-closed and $A \underset{B}{\perp} C$, etc (from last lecture).

Claim: $dcl(bdd(AB) \cup bdd(AC)) \cap bdd(BC) = dcl(BC)$.

Proof of claim: $\supseteq$ clear.

Assume $\alpha \in$ intersection. Assume α is a small hyperimaginary.

Then $\exists\ a \in A,\ b \in B,\ c \in C,\ \beta \in bdd(ab),\ \gamma \in bdd(ac)$, st. $\alpha \in dcl(\beta, \gamma) \cap bdd(bc)$, and we may take them to be small.

[If $\alpha \in \mathrm{bdd}(BC)$, let $q = tp(\alpha/BC)$, then for every $\varphi(x, x')$ contradicting $x = x'$, the type $\bigwedge_{i<\omega} q(x_i) \wedge \bigwedge_{i<j<\omega} \varphi(x_i, x_j)$ is contradictory, and one only needs finitely many parameters in BC for that.]

Since $A \underset{M}{\downarrow} BC$ and M is $|T|^+$-saturated, then ~~[illegible]~~. $a \underset{M}{\downarrow} \alpha bc$ so $\exists a' \in M$ s.t. $a' \equiv_{\alpha, b, c} a$ by "cover" property

ie $\exists \beta', \gamma'$ s.t. $\beta' \in \underbrace{\mathrm{bdd}(a', b)}_{\subseteq\, \mathrm{bdd}(B) = B}$, $\gamma' \in \underbrace{\mathrm{bdd}(a', c)}_{\subseteq\, \mathrm{bdd}(C) = C}$, $\alpha \in \mathrm{dcl}(\beta', \gamma')$.

so $\alpha \in \mathrm{dcl}(BC)$. claim $\square$

Now let $d \in \mathrm{bdd}(BC)$. ~~I claim that $tp(d/BC) \vdash$~~ ~~$tp(d/\mathrm{bdd}(AB) \cup$~~

Claim: $tp(d/BC) \vdash tp(d/\mathrm{bdd}(AB) \cup \mathrm{bdd}(AC))$.

Proof of Claim: ~~[illegible]~~.

let e be a code for the set of $[\mathrm{bdd}(AB) \cup \mathrm{bdd}(AC)]$-conjugates of d.

Then on the one hand, e codes a set of elements in $\mathrm{bdd}(BC) \Rightarrow e \in \mathrm{bdd}(BC)$.

On the other hand: $e \in dcl(bdd(AB) \cup bdd(AC))$.

$\Rightarrow$ $e \in dcl(BC)$ by claim.

$\Rightarrow$ what we wanted: if $d' \underset{BC}{\equiv} d \Rightarrow d' \underset{e}{\equiv} d \Rightarrow d' \in$ set that e codes. (claim)

Now we have $d \in bdd(BC)$, $e \in bdd(AB)$, $f \in bdd(AC)$ & want to show $def \equiv \sigma_{BC}(d)\, \sigma_{AB}(e)\, \sigma_{AC}(f)$.

We said we know that $\sigma_{AB} \cup \sigma_{AC}$ is elementary and therefore extends to an automorphism σ'.

let σ'' be $\sigma'^{-1} \circ \sigma_{BC}$.

Reduced to: $def \equiv \sigma''(d) ef$. ie $d \underset{ef}{\equiv} \sigma''(d)$.

But $\sigma' \supseteq \sigma_B$ & σ_C $\Rightarrow$ $\sigma''|_{B \cup C} = id$. $\Rightarrow d \underset{BC}{\equiv} \sigma''(d)$

by what we just said $\Rightarrow$ $d \underset{bdd(AB) \cup bdd(AC)}{\equiv} \sigma''(d) \Rightarrow d \underset{ef}{\equiv} \sigma''(d)$. $\square$

Where this leads:

"Knowing" $(bdd(AB), \sigma_{AB})$ ~~means~~ $\Rightarrow$ knowing $tp(AB)$ in the sense of T_A (where $\sigma_{AB} = \sigma\restriction_{bdd(AB)}$).

More generally, ~~tp^{T_A}~~ $tp^{T_A}(a)$ = automorphism type of $(\mathrm{bdd}(\sigma^{\mathbb{Z}}(a)), \sigma)$.

$$\mathrm{bdd}_\sigma(a) = \mathrm{bdd}^{T_A}(a).$$

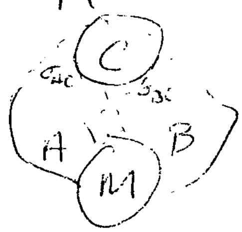
Want to define $a \underset{c}{\overset{\sigma}{\downarrow}} b$ if $\mathrm{bdd}_\sigma(ac) \underset{\mathrm{bdd}_\sigma(c)}{\downarrow} \mathrm{bdd}_\sigma(bc)$

Then assume $a \underset{M}{\overset{\sigma}{\downarrow}} b$ & have $c_1 \underset{M}{\overset{\sigma}{\downarrow}} a$ & $c_2 \underset{M}{\overset{\sigma}{\downarrow}} b$ &

$c_1 \underset{M}{\equiv} c_2$.

Write $A := \mathrm{bdd}_\sigma(a, M)$ etc so $A \underset{M}{\downarrow} B$, $C_1 \underset{M}{\downarrow} A$, $C_2 \underset{M}{\downarrow} B$.

Then we find a new C st. $C \downarrow AB$



$\dashrightarrow$ $C \overset{T_A}{\underset{AM}{\equiv}} C_1$ & $C \overset{T_A}{\underset{BM}{\equiv}} C_2$

so $C \underset{M}{\overset{\sigma}{\downarrow}} AB$ & proved ind. thm for M.