Chapter 42

Handbook of Jacobians

Function	Sizes	Jacobian
$Y = BXA^T$ $= (A \bigotimes B)X$ "Kronecker Product"	$X \quad m \times n$ $A \quad n \times n$ $B \quad m \times m$	$(\mathrm{d}Y) = (\det A)^n (\det B)^m$ 8.3.1
$Y = X^{-1}$ "inverse"	$X \quad n \times n$	$(\mathrm{d}Y) = (\det X)^{-2n} (\mathrm{d}X)$
$Y = X^2$	$X \quad n \times n$	$(\mathrm{d}Y) = \prod_{i,j} \lvert\lambda_i + \lambda_j\rvert (\mathrm{d}X)$
$Y = X^k$	$X \quad n \times n$	$(\mathrm{d}Y) = \prod_{i \geq j} \left\lvert \sum_{l=0}^{k-1} \lambda_i^l \lambda_j^{n-1-l} \right\rvert (\mathrm{d}X)$ 8.3.2
$Y = \frac{1}{2}(AXB + B^T XA)$ "Symmetric Kronecker"	$X, A, B \quad n \times n$ $A, B \quad \text{sym}$ $B = A^T$	$(\mathrm{d}Y) = \prod_{i \leq j} \lvert\lambda_i M_j + \lambda_j M_i\rvert (\mathrm{d}X)$ $(\mathrm{d}Y) = (\det A)^{n+1} (\mathrm{d}X)$

42.1 Real Case; General Matrices

In factorizations that involve a square $m \times m$ orthogonal matrix Q, and start from a thin, tall $m \times n$ matrix A ($m \geq n$ *always*), $H^T \mathrm{d}Q$ stands for the wedge product

$$H^T \mathrm{d}Q = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n} q_i^T \mathrm{d}q_j \,.$$

Here H is a stand-in for the first n columns of Q. If the differential matrix and the matrix whose transpose we multiply by are one and the same, the wedging is done over all of the columns. For example,

$$V^T \mathrm{d}V = \bigwedge_{i=1}^{n} \bigwedge_{j=i+1}^{n} v_i^T \mathrm{d}v_j,$$

for V an $n \times n$ orthogonal matrix.

The Jacobian of the last factorization (non-symmetric eigenvalue) is computed only for the positive-measure set of matrices with real eigenvalues and with a full set of eigenvectors.

Factorization	Matrix sizes & properties	Parameter count	Jacobian
$A = LU$ lu	$A, L, U \;\; n \times n$ L, U^T lower triangular $l_{ii} = 1, \; \forall i = 1, n$	$n^2 =$ $\frac{n(n-1)}{2} + \frac{n(n+1)}{2}$	$(dA) =$ $\prod_{i=1}^{n} u_{ii}^{n-i} (dL)(dU)$
$A = QR$ qr	$A, R \;\; m \times n$ $Q \;\; m \times m$ $Q^T Q = I_n$ R upper triangular	$mn =$ $\left(mn - \frac{n(n+1)}{2}\right) + \frac{n(n+1)}{2}$	$(dA) =$ $\prod_{i=1}^{n} r_{ii}^{m-i} (dR)(H^T dQ)$
$A = U \Sigma V^T$ svd	$A, \Sigma \;\; m \times n$ $U \;\; m \times m, \; V \;\; n \times n$ $U^T U = I_m$ $V^T V = I_n$	$mn =$ $\left(mn - \frac{n(n+1)}{2}\right) + n + \frac{n(n-1)}{2}$	$(dA) =$ $\prod_{i<j} (\sigma_i^2 - \sigma_j^2) \prod_{i=1}^{n} \sigma_i^{m-n} (d\Sigma)(H^T dU)(V^T dV)$
$A = QS$ polar	$A, Q \;\; m \times n$ $S \;\; n \times n$ $Q^T Q = I_n$ S positive definite	$mn =$ $\left(mn - \frac{n(n+1)}{2}\right) + \frac{n(n+1)}{2}$	$(dA) =$ $\prod_{i<j} (\sigma_i + \sigma_j)(dS)(Q^T dQ)$
$A = X \Lambda X^{-1}$ nonsymm eig	$A, X, \Lambda \;\; n \times n$ Λ diagonal	$n^2 =$ $(n^2 - n) + n$	$(dA) =$ $\prod_{i<j} (\lambda_i - \lambda_j)^2 (d\Lambda)(X^{-1} dX)$

42.2 Real Matrices; Symmetric "+" Cases

Here we gather factorizations that are symmetric or, in the case when this is required, positive definite. All matrices are square $m \times m$.

Factorization	"+"	Parameter count	Matrix properties	Jacobian
$A = LL'$ Choleski	Positive definite	$\frac{n(n+1)}{2} =$ $\frac{n(n+1)}{2}$	L lower triangular	$(dA) =$ $2^n \prod_{i=1}^{n} l_{ii}^{n+1-i} (dL)$
$A = LDL'$ ldl	—	$\frac{n(n+1)}{2} =$ $\frac{n(n-1)}{2} + n$	L lower triangular $l_{ii} = 1, \; \forall i = 1, n$ D diagonal	$(dA) =$ $\prod_{i=1}^{n} d_i^{m-i} (dL)(dD)$
$A = Q \Lambda Q^T$ eig	—	$\frac{n(n+1)}{2} =$ $\frac{n(n-1)}{2} + n$	$Q^T Q = I_n$ Λ diagonal	$(dA) =$ $\prod_{i<j} \lvert \lambda_i - \lambda_j \rvert (d\Lambda)(Q^T dQ)$

42.3 Real Matrices; Orthogonal Case

This is the case of the CS decomposition. Note that the matrices U_1 and V_1 share $\frac{p(p-1)}{2}$ parameters; this is due to the fact that the product of the first p columns of U_1 and the first p rows of V_1^T is invariant and determined. This is equivalent to saying that we introduce an equivalence relation of the set of pairs (U_1, V_1) of orthogonal matrices of size k, by having

$$(U_1, V_1) \sim \left(U_1 \begin{bmatrix} Q & 0 \\ 0 & I_j \end{bmatrix}, V_1 \begin{bmatrix} Q & 0 \\ 0 & I_j \end{bmatrix} \right) \; ,$$

for any $p \times p$ orthogonal matrix Q.

Since it is rather hard to integrate over a manifold of pairs of orthogonal matrices with the equivalence relationship mentioned above, we will use a different way of thinking about this splitting of parameters. We can assume that U_1 contains $\frac{k(k-1)}{2}$ parameters (a full set) while V_1 only contains $\frac{k(k-1)}{2} - \frac{p(p-1)}{2}$ (having the first p columns already determined from the relationship with U_1). Alternatively, we could assign a full set of parameters to V_1 and think of U_1 as having the first r columns predetermined. In the following we choose to make the former assumption, and think of the undetermined part H of V_1^T as a point in the Stiefel manifold $V_{j,k}$.

Factorization	Matrix sizes & properties	Parameter count	Jacobian
$Q =$ $\begin{bmatrix} U_1 & 0 \\ 0 & U_2 \end{bmatrix} \begin{bmatrix} I_p & 0 & 0 \\ 0 & C & S \\ 0 & S & -C \end{bmatrix} \begin{bmatrix} V_1 & 0 \\ 0 & V_2 \end{bmatrix}^T$ $n = k + j,\ p = k - j$ CS decomposition	$Q\ \ n \times n$ $U_1, V_1\ \ k \times k$ $U_2, V_2\ \ j \times j$ $Q^TQ = I_n$ U_1, V_1, U_2, V_2 orthogonal $C = \mathrm{diag}(\cos(\theta_i)),\ i = 1 \dots n$ $S = \mathrm{diag}(\sin(\theta_i)),\ i = 1 \dots n$	$U_1V_1^T$ has $k(k-1) - \frac{p(p-1)}{2}$ U_2, V_2 have $\frac{j(j-1)}{2}$ each C and S share j	$(dA) =$ $\prod_{i<j} sin(\theta_j - \theta_i) sin(\theta_i + \theta_j)(d\theta)$ $(U_1^T dU_1)(U_2^T dU_2)(H^T dH)(V_2^T dV_2)$

42.4 Real Matrices; Symmetric Tridiagonal

The important factorization of this section is the eigenvalue decomposition of the symmetric tridiagonal.The tridiagonal is defined by its $2n - 1$ parameters, n on the diagonal and $n - 1$ on the upper diagonal:

$$T = \begin{pmatrix} a_n & b_{n-1} & & & \\ b_{n-1} & a_{n-1} & b_{n-2} & & \\ & \ddots & \ddots & \ddots & \\ & & b_2 & a_2 & b_1 \\ & & & b_1 & a_1 \end{pmatrix}.$$

The matrix of eigenvectors Q can be completely determined from knowledge of its first row $q = (q_1, \dots, q_n)$ $(n-1$ parameters) and the eigenvalues on the diagonal of Λ (another n parameters). Here dq represents integration over the $(1, n)$ Stiefel manifold.

Factorization	Matrix sizes	Parameter count	Matrix properties	Jacobian
$T = Q\Lambda Q^T$ eig	$T, Q, \Lambda\ \ n \times n$	$2n - 1 =$ $(n-1) + n$	$Q^TQ = I_n$ Λ diagonal	$(dT) = \frac{\prod_{i=1}^{n-1} b_i}{\prod_{i=1}^{n} q_i}(d\Lambda)(dq)$

Chapter 43

Joint Densities

Table of Joint Element Densities

General Matrices	MATLAB	Joint element density
$G(m,n)$	$\texttt{randn}(m,n)$	$(2\pi)^{-mn/2}\exp(-\frac{1}{2}A^TA)$
$G(n,n)$	$\texttt{randn}(n)$	$(2\pi)^{-n^2/2}\exp(-\frac{1}{2}A^TA)$
$G(m,n;A,B\otimes A)$	$B^{1/2}\,\texttt{randn}(m,n)\cdot {A^{-1/2}}^T + M$	
$G(m,n;B,M)$	$B^{1/2}\,\texttt{randn}(m,n) + M$	
$G(m,n;B)$	$B^{1/2}\,\texttt{randn}(m,n)$	

Symmetric Matrices	MATLAB	Joint element density
GUE	$A = \texttt{randn}(n); S = (A + A')/2$	$2^{-n/2}\pi^{-\frac{n(n+1)}{4}}\exp(-\frac{1}{2}S^2)$
$W(m,n)$	$A = \texttt{randn}(m,n); S = A^TA$	$2^{-mn/2}\Gamma_n^{-1}(\frac{m}{2})\lvert S\rvert^{(m-n-1)/2}\exp(-\frac{1}{2}S)$
$W(m,n;\Sigma)$	$A = \texttt{randn}(m,n); S = A^T\Sigma A$	$2^{-mn/2}\Gamma_m^{-1}(\frac{m}{2})\lvert\Sigma\rvert^{-m/2}\lvert S\rvert^{(m-n-1)/2}\exp(-\frac{1}{2}\Sigma^{-1}S)$
$W(m,n;\Sigma;M)$	$A = \Sigma^{1/2}\,\texttt{randn}(m,n)\cdot M; S = A^TA$	