18.969 Topics in Geometry: Mirror Symmetry
Spring 2009

MIRROR SYMMETRY: LECTURE 10

DENIS AUROUX

1. The Quintic (contd.)

Recall that we had a quintic mirror family $\check{X}_\psi$ with LCSL degeneration as $z = (5\psi)^{-5} \to 0$. We had the Picard-Fuchs equation for periods of $\check{\Omega}$, and found 2 solutions given by

$$
\begin{aligned}
&\phi_0(z) = \sum_{n=0}^{\infty} \frac{(5n)!}{(n!)^5} z^n \\
&\phi_1(z) = \phi_0(z)\log z + \tilde{\phi}(z), \tilde{\phi}(z) = 5\sum_{n=1}^{\infty} \frac{(5n)!}{(n!)^5}\left(\sum_{j=n+1}^{5n}\frac{1}{j}\right) z^n
\end{aligned}
\tag{1}
$$

We then obtained canonical coordinates $\beta_0, \beta_1 \in H_3(\check{X}, \mathbb{Z})$ for the complex moduli space s.t. the monodromy preserves β_0 and maps $\beta_1 \mapsto \beta_1 + \beta_0$, and $\int_{\beta_i}\check{\Omega}$ are linear combinations of ϕ_0, ϕ_1. We wrote

$$
w = \frac{\int_{\beta_1}\check{\Omega}}{\int_{\beta_0}\check{\Omega}}, q = \exp(2\pi i w) = c_2 z \exp\left(\frac{\tilde{\phi}(z)}{\phi_0(z)}\right)
\tag{2}
$$

where c_2 is a normalization constant.

1.1. Yukawa coupling on $H^{2,1}(\check{X})$.

Let

$$
W_k = \int_{\check{X}_z} \check{\Omega}(z) \wedge \frac{d^k}{dz^k}\check{\Omega}(z)
\tag{3}
$$

We can rewrite the Picard-Fuchs equation in the form

$$
\frac{d^4}{dz^4}[\check{\Omega}] + \sum_{k=0}^{3} c_k(z)\frac{d^k}{dz^k}[\check{\Omega}] = 0
\tag{4}
$$

Then $W_4 + \sum_{k=0}^{3} c_k W_k = 0$. By Griffiths transversality ($\frac{d^k}{dz^k}\check{\Omega}$ has no $(0,3)$-component unless $k \geq 3$), $W_0 = W_1 = W_2 = 0$. Moreover,

$$\begin{aligned} 0 = \frac{d^2}{dz^2}W_2 &= \int_{\check{X}} \frac{d^2\check{\Omega}}{dz^2} \wedge \frac{d^2\check{\Omega}}{dz^2} + 2\int_{\check{X}} \frac{d\check{\Omega}}{dz} \wedge \frac{d^3\check{\Omega}}{dz^3} + \int_{\check{X}} \check{\Omega} \wedge \frac{d^4\check{\Omega}}{dz^4} \\ &= 0 + 2\left(\frac{dW_3}{dz} - W_4\right) + W_4 \end{aligned} \tag{5}$$

implying that $W_4 = 2W_3'$, hence $W_3'(z) = -\frac{1}{2}c_3(z)W_3(z)$. Looking at the coefficients on the Picard-Fuchs equation gives

$$\begin{aligned} c_3(z) = \frac{6}{z} - \frac{2 \cdot 5^5}{1 - 5^5 z} &\implies (\log W_3') = \frac{-3}{z} + \frac{5^5}{1-5^5 z} \\ &\implies W_3(z) = \frac{c_1}{(2\pi i)^3 z^3 \cdot (5^5 z - 1)} \end{aligned} \tag{6}$$

We next normalize $\check{\Omega}$: scaling by $f(z)$ changes

$$\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle = \int f\check{\Omega} \wedge \left(\frac{d^3}{dz^3} f\check{\Omega}\right) = f^2 \int \check{\Omega} \wedge \frac{d^3\check{\Omega}}{dz^3} \tag{7}$$

We want to scale by $\frac{1}{\int_{\beta_0}\check{\Omega}} = \frac{\text{const}}{\phi_0(z)}$, giving

$$\langle \frac{\partial}{\partial z}, \frac{\partial}{\partial z}, \frac{\partial}{\partial z} \rangle = \frac{c_1}{(2\pi i)^3 z^3 \cdot (5^5 z - 1)\phi_0(z)^2} \tag{8}$$

Switching to $\frac{\partial}{\partial w} = (\frac{dw}{dz})^{-1}\frac{\partial}{\partial z}$ gives us

$$\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = \frac{c_1}{\cdot (5^5 z - 1)\phi_0(z)^2 \delta(z)^3} \tag{9}$$

where

$$\delta(z) = 2\pi i z \frac{dw}{dz} = z\frac{d}{dz}(\log q) = 1 + z\frac{d}{dz}\left(\frac{\tilde{\phi}(z)}{\phi_0(z)}\right) \tag{10}$$

To express this as a power series in q:

$$\begin{aligned} \frac{dq}{dz} = q\frac{d\log q}{dz} = \frac{q}{z}\delta(z) &= c_2\delta(z)\exp\left(\frac{\tilde{\phi}(z)}{\phi_0(z)}\right) \\ \frac{d^j}{dq^j}\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle &= \left(\frac{1}{c_2\delta(z)\exp(\tilde{\phi}/\phi_0)}\frac{d}{dz}\right)^j \left(\frac{c_1}{\cdot (5^5 z - 1)\phi_0(z)^2\delta(z)^3}\right) \end{aligned} \tag{11}$$

Solving and expanding out, we obtain

$$\langle \frac{\partial}{\partial w}, \frac{\partial}{\partial w}, \frac{\partial}{\partial w} \rangle = -c_1 - 575\frac{c_1}{c_2}q - \frac{1950750}{2}\frac{c_1}{c_2^2}q^2 - \frac{10277490000}{6}\frac{c_1}{c_2^3}q^3 + \cdots \tag{12}$$

Now we can describe the mirror symmetry: there exists a basis of $H^2(X,\mathbb{Z}) \cong \mathbb{Z}$ (where X is the original quintic) given by the Poincaré dual $\{e\}$ of a hyperplane s.t., writing $[B+i\omega] = te, q = \exp(2\pi i t) = \exp(2\pi i \int_{line} B + i\omega)$, the mirror map is

$$q \leftrightarrow q, w = \frac{1}{2\pi i}\log q \leftrightarrow t, \frac{\partial}{\partial w} \leftrightarrow \frac{\partial}{\partial t} = e \tag{13}$$

where the latter correspondence is how we match $H^{2,1}(\check{X}) = T\mathcal{M}_{cx}(\check{X}) \cong T\mathcal{M}_{Kah}(X) \cong H^{1,1}(X)$. Recall that

$$\langle e,e,e\rangle = \int_X e \wedge e \wedge e + \sum_{d>0} \langle e,e,e\rangle_{0,d} q^d \tag{14}$$

where $\langle e,e,e\rangle_{0,d}$ is the Gromov-Witten invariant $(\int_d e)(\int_d e)(\int_d e)N_d = d^3 N_d$ of degree d rational curves through three general hyperplanes, and $N_d = \sum_{d=kd'} \frac{n_{d'}}{k^3}$ counts multiple covers. Expanding out, we obtain

(15)

$$\begin{aligned}\langle e,e,e\rangle &= 5 + \sum_{d>0} d^3 N_d q^d = 5 + \sum_{d>0} d^3 n_3 \frac{q^d}{1-q^d}\\ &= 5 + n_1 q + 8\left(n_2 + \frac{n_1}{8}\right) q^2 + 27\left(n_3 + \frac{n_1}{27}\right) q^3 + 64\left(n_4 + \frac{n_2}{8} + \frac{n_1}{64}\right) q^4 + \cdots\end{aligned}$$

Matching these gives

- $c_1 = -5$.
- $n_1 = \frac{575\cdot 5}{c_2} = \frac{2875}{c_2}$: classical algebraic geometry tells us that 2875 is the number of lines on a quintic, $c_2 = 1$.
- $n_2 = 609250$ (had been calculated by Sheldon Katz, 1986)
- $n_3 = 317206375$ (Ellingsrud-Stromme, 1990)
- $n_4 = 242467530000$

The general verification is in the proof of mirror symmetry for the quintic by Givental and Lian-Liu-Yau separately around 1996 (more generally, they verify for Calabi-Yau complete intersections in toric varieties).

2. Homological Mirror Symmetry

This is a different mathematical formulation of mirror symmetry, given by Kontsevich in 1994. On the symplectic side, just as J-holomorphic curves gave a "quantum" intersection product on $H^*(X)$, we will look at intersections of Lagrangian submanifolds, and obtain a "quantum" intersection theory involving J-holomorphic disks. On the complex side, we look at intersections of subvarieties and holomorphic maps/extensions of bundles/sheaves. Thus, the complex side is governed by "classical" algebraic geometry, and all the "quantum" information is on the symplectic side. For this, we will construct the Fukaya (A_∞)-category, which is roughly the category whose objects are Lagrangian submanifolds, whose morphisms are intersections, and whose algebraic structures (differential, product, etc.) are governed by J-holomorphic disks. On the complex side, we just have coherent sheaves, and our mirror symmetry will give an equivalence of derived categories.

Future question: what is the relationship between this form of mirror symmetry and our previous one? Basic answer: open string theory gave an idea of considering submanifolds with boundary lying on branes. Kontsevich himself looked at the Hochschild cohomologies of the two categories above, which give the "big" quantum cohomology and the cohomology ring of polyvector fields on the respective sides.

2.1. **Lagrangian Floer Homology.** Let (M,ω) be a symplectic manifold, L_0, L_1 compact Lagrangian submanifolds. Assume that L_0, L_1 intersect transversely, i.e. $L_0 \cap L_1$ is a finite set. Recall that we defined the Novikov ring as $\Lambda = \{\sum a_i T^{\lambda_i} \mid \lambda_i \to \infty\}$. The *Floer complex* $CF(L_0, L_1)$ is the free Λ-module $\Lambda^{|L_0 \cap L_1|}$ generated by $L_0 \cap L_1$. Our goal is to define a differential δ s.t. $HF(L_0, L_1) = H^*(CF, \delta)$ is invariant under Hamiltonian isotopies. The motivation for this was to understand Arnold's conjecture on Lagrangian intersections. From that point of view, HF is an obstruction to displacement of Lagrangians: in general, if we have a topological isotopy between two Lagrangian submanfiolds, a pair of intersections can be cancelled along a *Whitney disk* (its corners are the intersections of the two Lagrangian submanifolds; Hamiltonian isotopies cancel intersection along holomorphic Whitney disks. δ will count holomorphic disks M between Lagrangian submanifolds.