18.969 Topics in Geometry, MIT Fall term, 2006

Problem sheet 4

Exercise 1. Let L, L' be transverse almost Dirac structures, inducing a $\mathbb{Z}$-grading on the spinors

$$S = \oplus_{k=0}^{n} \mathcal{U}_k,$$

where $\mathcal{U}_k = \wedge^k L^* \cdot K_L$, using the identification $L^* = L'$. As we saw, the twisted derivative $d = d_H$ acting on $\mathcal{U}_k$ decomposes as $d = (\pi_{k-3}+\pi_{k-1}+\pi_{k+1}+\pi_{k+3})\circ d$. Show that the operators $T = \pi_{k+3} \circ d$, $T' = \pi_{k-3} \circ d$ are given by the Clifford action of tensors $T \in \wedge^3 L^*$ and $T' \in \wedge^3 L$, respectively, where

$$T(a, b, c) = \langle [a, b], c\rangle$$

and similarly for T'. Hence conclude that the integrability of L, L' may be expressed as the vanishing of the tensors T, T' respectively.

Exercise 2. Let L be a Dirac structure and let L' be a transverse almost dirac structure. Use the derived bracket formalism to show that

$$(d_L x)\cdot = [\partial, x\cdot]$$

as operators on $S = \Omega^\bullet(M)$, where $x \in C^\infty(\wedge^\bullet L^*)$ and $\partial = \pi_{k-1} \circ d$ on $\mathcal{U}^k$ in the notation of the previous problem.

Exercise 3. Let L, L' be transverse Dirac structures (both integrable). Show that

$$d_*[X, Y] = [d_* X, Y] + [X, d_* Y],$$

where $X, Y \in C^\infty(L)$, $[\cdot,\cdot]$ is the induced Lie bracket on L, and d_* is the Lie algebroid differential for $L' = L^*$. Hint: Use the Jacobi identity for the Courant bracket and the description of d_* via

$$(d_* X)\cdot = [\partial, X\cdot]$$

as operators on S.

Exercise 4. Let L be the complex Dirac structure associated to a generalized complex structure $\mathcal{J}$, and let K_L be the complex pure spinor line defining L. Use the Mukai pairing to demonstrate that

$$2c_1(K_L) = c_1^+ + c_1^-,$$

where $c_1^\pm$ are the first Chern classes of the $U(n, n)$ structure defined by $\mathcal{J}$. Explain why $c_1^+ + c_1^-$ must be even *a priori*.

Exercise 5. Let $\mathcal{J}$ be a generalized complex structure on the exact Courant algebroid E such that $\mathcal{J}T^* = T^*$. Write the decomposition of $\mathcal{J}$ given a general (non-complex) splitting $s : T \longrightarrow E$. Hint: determine the difference between the splittings s and $-\mathcal{J}sJ$, where J is the induced complex structure on $E/T^* = T$. How does this compare to the expression of $\mathcal{J}$ in a complex splitting?

Exercise 6. Let $\mathcal{J}$ be an almost generalized complex structure. Show that

$$\mathcal{N}_{\mathcal{J}}(x, y) = [\mathcal{J}x, \mathcal{J}y] - \mathcal{J}[\mathcal{J}x, y] - \mathcal{J}[x, \mathcal{J}y] - [x, y]$$

is tensorial, and express it in terms of the tensors T, T' from the first exercise.

Exercise 7. Let $\mathcal{J}$ be a generalized complex structure. Show that $e^{\theta\mathcal{J}}(T^*)$ is a Dirac structure for all θ.

Exercise 8. Let $\varphi = e^{B+i\omega}\Omega$ be the complex pure spinor corresponding to a generalized complex structure $\mathcal{J}$, and let $f : \Delta \longrightarrow T$ be the inclusion of the symplectic foliation determined by $\mathcal{J}$ associated with the canonical Poisson structure π. Show that $f^*\omega$ coincides with the symplectic form induced by π on Δ.

Exercise 9. Let (g, I, J) define a hyperKähler structure, so that (g, I), (g, J) are Kähler and $IJ = -JI =: K$. Let $\omega_I, \omega_J, \omega_K$ be the associated symplectic forms. Verify that for a, b, c real and $a^2 + b^2 + c^2 = 1$,

$$\mathcal{J}(a, b, c) = a\mathcal{J}_I + b\mathcal{J}_{\omega_J} + c\mathcal{J}_{\omega_K}$$

squares to -1, and is an orthogonal endomorphism of $T \oplus T^*$. Also prove that for $a \neq 0$, $\mathcal{J}(a, b, c)$ is a B-field transform of a symplectic structure. Conclude that $\mathcal{J}(a, b, c)$ is an integrable generalized complex structure for all points on the sphere.