11 The embedding manifolds in $\mathbb{R}^N$

Theorem 11.1. *(The Whitney Embedding Theorem, Easiest Version). Let X be a compact n-manifold. Then X admits a embedding in $\mathbb{R}^N$.*

Proof. First we construct an embedding $\Phi : X \to \mathbb{R}^N$ for some large N. Let $\{f_i\}_{i=1}^k$ be a partition of unity so that the support of each f_i is contained in some coordinate chart (U_i, ϕ_i) so that $\phi_i(U_i)$ is bounded. Then we can construction smooth functions $\tilde{\phi}_i : X \to \mathbb{R}^n$ by

$$\tilde{\phi}_i(x) = \begin{cases} f_i(x)\phi_i(x) & \text{if} \quad x \in U_i \\ 0 & \text{if} \quad x \in U_i \end{cases} .$$

Then we can define Φ by the equation

$$\Phi(x) = (\tilde{\phi}_1(x), \tilde{\phi}_2(x), \dots, \tilde{\phi}_k(x),\ f_1(x),\ f_2(x), \dots,\ f_k(x)).$$

Then $\Phi(x) = \Phi(x')$ implies that for some i, $f_i(x) = f_i(x') \neq 0$ so that $x, x \in U_i$. Then for the same i we have

$$\phi_i(x) = \phi_i(x')$$

and hence $x = x'$ since ϕ_i is a diffeomorphism on U_i and so Φ is injective.

Next we need to check that the differential of Φ is injective. The differential of Φ at x send $v \in T_x X$ to

$$(D_x f_1(v)\phi_1(x) + f_1(x)D_x\phi_1(v), \dots, D_x f_k(v)\phi_k(x) + f_k(x)D_x\phi_k(v), D_x f_1(v), \dots, D_x f_k(v))$$

and the result follows. □