Lecture 30.

21.2 The Frobenious Integrability Theorem

Next we consider when can a subbundle ξ of the tangent bundle TM of M can be brought into a canonical form. In generality this is a very complicated problem and we need to isolate manageable cases. The example that comes to mind is the case where $\Xi_0|_{(x,y)} = T_x\mathbb{R}^n \times \{0\} \subset T_x\mathbb{R}^n \times T_x\mathbb{R}^{m-n}$, the tangent bundle along a product. A subbundle which is locally diffeomorphic to Ξ_0 is called *integrable*.

Notice that Ξ_0 is has following property. If

$$X_1 = \sum_{i=1}^{n} a^i(x^1, \ldots, x^m)\frac{\partial}{\partial x^i}, \quad \text{and} \quad X_2 = \sum_{i=1}^{n} b^i(x^1, \ldots, x^m)\frac{\partial}{\partial x^i}$$

is a pair of local sections of Ξ_0 then the bracket

$$[X_1, X_2] = \sum_{i,j=1}^{n} \big(a^i \frac{\partial b^j}{\partial x^i} - b^i \frac{\partial a^j}{\partial x^i}\big)\frac{\partial}{\partial x^j}$$

is also a local section of Ξ. A subbundle with this property is called *involutive*. Clearly any integrable subbundle is involutive.
Examples:

$$\Xi_1 = \text{span}\{\frac{\partial}{\partial x} + \frac{2zx}{1+x^2+y^2}\frac{\partial}{\partial z}, \frac{\partial}{\partial y} + \frac{2zy}{1+x^2+y^2}\frac{\partial}{\partial z}\}$$

is involutive indeed it field of tangent planes to the family of paraboloids

$$z = \lambda(1 + x^2 + y^2)$$

On the other hand

$$\Xi_2 = \text{span}\{\frac{\partial}{\partial x} + y\frac{\partial}{\partial z}, \frac{\partial}{\partial y}\}$$

is not involutive. In fact in has the interesting property that given any two points and any path connected neighborhood there is a path tangent to Ξ_2 joining the two points contained in the neighborhood. Clearly then Ξ_2 is not integrable.

The following provides a converse.

Theorem 21.4. *(Frobenius). If Ξ is involutive then it is integrable.*

Proof. Choose first a coordinate patch about of the from $\phi : U \to \mathbb{R}^n \times \mathbb{R}^{m-n}$ so that at $\phi(m) = 0$ and $\phi_*(\xi_m) = T_0\mathbb{R}^n \times \{0\}$. Set $\Xi_1 = \phi_*(\Xi)$.

Then in some neighborhood $V \times W$ of $\phi(m) = 0$ we can find a function $f : V \times W \times \mathbb{R}^n \to \mathbb{R}^{m-n}$, linear in the last factor with $f(0, 0, \cdot) = 0$ and so that any $\xi \in \Xi$ can uniquely be written as

$$\xi = (e,\ f(x, y, e).$$

There is a natural homotopy of Ξ_0 to Ξ_1 given by

$$\Xi_t = \{(e, tf(tx, y, e)|e \in \mathbb{R}^n\}.$$

We will show that there is a one parameter family of diffeomorphisms F_t so that

1. $F_t(0) = 0$ and
2. $(F_t)_*(X_t) = \Xi_0$.

Thus F_1 is the desired change of coordinates. For $x \in V$ let

$$X_x(v, w) = (x,\ f(v, w, x))$$

Then the fact the Ξ_1 is involutive implies that $[X_x, X_y] \in \Xi_1$ but $[X_x, X_y]$ is certainly of the form $(O, *)$ since the constant vectors fields x and y commute so $[X_x, X_y] = 0$. More explicitly

$$[X_x, X_y] = \big(0, D_{(v,w,x)}f(y,\ f(v, w, y), 0) - D_{(v,w,y)}f(x,\ f(v, w, x), 0)\big) = 0$$

Let $X_t(v, w) = (0, f(tv, w, v))$. A typical section of Ξ_t is $X_{t,x}(u, v) = (x, tf(tv, w, x))$. We can work out the bracket $[X_t, X_{t,x}]$

$$\begin{aligned}
[X_t, X_{t,x}] &= \big(0, tD_{(tv,w,x)} f(0, f(tv, w, v), 0) \\
&\quad -tD_{(tv,w,v)} f(x, f(tv, w, x), 0) - f(tv, w, x)\big) \\
&= -tD_{(tv,w,x)} f(v, 0, 0) - f(tv, w, x) \\
&= -\frac{d}{dt} X_{t,x}
\end{aligned}$$

Thus the Lie derivative of $[(X_t, \frac{d}{dt}), X_{t,x}] = 0$ or equivalently if F_t is the flow of the time dependent vector field then we have $(F_t)_*(X_{s,x}) = X_{s+t,x}$ as required. □

Here is a more intuitive proof by induction on the dimension.

Proof. Induction on the dimension of the subbundle. The case of dimension one follows from the standard form for an non-vanishing vector field. The question is also local so we assume that we are given a subbundle of the tangent bundle of $\mathbb{R}^n$ defined in a neighborhood of $0 \in \mathbb{R}^n$. Suppose we have proved the result for all subbundles of dimension d. Let E be an involutive subbundle of $T\mathbb{R}^n$ of dimension $d+1$. Choose a nowhere vanishing local section, X, of E. Next choose a coordinate system $z^1, \dots, z^n$, centered at 0, so that $\frac{\partial}{\partial z^n} = X$. $T\mathbb{R}^{n-1} \times \{0\}$ is an integrable hence involutive subbundle. $E' = E \cap T\mathbb{R}^{n-1} \times \{0\}$ defines a subbundle in a neighborhood of 0 of dimension d. Since E' is the intersection of two involutive subbundles it is involutive and so the induction hypothesis applies. We can find a coordinate system $y^1, \dots, y^n$ centered at 0 so that E' is given in a neighborhood of 0 as the span of $y^1, \dots, y^d$ In this new coordinate system X may not be straight but we have that

$$\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^d}, X$$

forms a basis for E. We can write

$$X = \sum_{i=1}^{d} a^i \frac{\partial}{\partial y^i} + X_0$$

where X_0 is section of TW. Then

$$\frac{\partial}{\partial y^1}, \dots, \frac{\partial}{\partial y^d}, X_0$$

is also a basis for E. Since X_0 is a section of TW so is $[\frac{\partial}{\partial y^i}, X_0]$. By involutivity it is parallel to X_0 so there is a smooth function f_1 defined in a neighborhood of 0 with

$$[\frac{\partial}{\partial y^i}, X_0] = f_1 X_0.$$

Set

$$g_1 = -int_0^{y^1} f_i(\mathbf{w}, s, y^2, \ldots, y^d) ds.$$

Then set

$$X_1 = \exp(g_1) X_0.$$

It is now easy to check that

$$[\frac{\partial}{\partial y^i}, X_1] = 0.$$

X_1 is still a section of TW so $[\frac{\partial}{\partial y^i}, X_0]$ is parallel to X_1 and we can find a smooth function f_2 so that

$$[\frac{\partial}{\partial y^i}, X_1] = f_2 X_1$$

We claim that

$$\frac{\partial f_2}{\partial y^1} = 0.$$

To see this notice that

$$[\frac{\partial}{\partial y^1}, [\frac{\partial}{\partial y^2}, X_1] = \frac{\partial f_2}{\partial y^1} X_1 = 0.$$

Using Jacobi's identity we also have

$$\begin{aligned} [\frac{\partial}{\partial y^1}, [\frac{\partial}{\partial y^2}, X_1] &= [[\frac{\partial}{\partial y^1}, \frac{\partial}{\partial y^2}] X_1] + [\frac{\partial}{\partial y^2}, [\frac{\partial}{\partial y^1}, X_1] \\ &= 0. \end{aligned}$$

So if we set

$$g_2 = -\int_0^{y^2} f_i(\mathbf{w}, y^1, s, y^3, \ldots, y^d) ds.$$

and

$$X_2 = e^{g_2} X_1$$

we have

$$[\frac{\partial}{\partial y^i}, X_2] = 0$$

for $i = 1, 2$. Continuing in this fashion we eventually find X_d commuting with $y^1, \ldots, y^d$ and we can construct the desired coordinate system as we did in class. $\square$

21.3 Foliations

The local structure of the previous subsection has as its global counterpart the notion of a foliation. Here is the precise definition.

Definition 21.5. A foliation $\mathcal{F}$ of M is a decomposition of M as a disjoint union of connected immersed submanifolds $M = \coprod_{\alpha \in A} \mathcal{L}_\alpha$ called the leaves of $\mathcal{F}$ so that each point has a chart (U, ϕ) so that under ϕ the decomposition obtained from the decomposition $\coprod_{\alpha \in A} \mathcal{L}_\alpha \cap U$ by taking components goes over to the decomposition of $\mathbb{R}^n = \coprod_{x \in \mathbb{R}^{n-k}} \mathbb{R}^k \times x$.

It is important to realize that in the above definition we do not require the leaves to have the subspace topology. For example Consider the 2-torus

$$T^2 = \mathbb{R}^2 / \mathbb{Z}^2$$

Fix a pair of real numbers (ζ_1, ζ_2) so that ζ_1/ζ_2 is irrational. The cosets of the subgroup Γ generated by $\{[t\zeta_1, t\zeta_2] | t \in \mathbb{R}\}$ give rise to a foliation with leaves that are not locally closed subsets.

Remark 6. The space of leaves of a foliation is one setting where one runs into non-Hausdorff manifolds. The space of leaves has a natural covering by charts (These may not be injective so be careful).

22 Characterizing a codimension one foliation in terms of its normal vector.

Let $\mathcal{F}$ be a two dimensional foliation of $\mathbb{R}^3$.

Proposition 22.1. *Let* $\mathbf{n}$ *be a local normal vector field to* $\mathcal{F}$. *Then*

$$\mathbf{n} \cdot (\nabla \times \mathbf{n}) = 0$$

Proof. Write
$$\mathbf{n} = a\frac{\partial}{\partial x} + b\frac{\partial}{\partial y} + c\frac{\partial}{\partial z}.$$
By rotating the coordinates we can assume that none of a, b or c are zero. Then $\mathcal{F}$ is locally spanned by the local sections
$$-b\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}, c\frac{\partial}{\partial x} - a\frac{\partial}{\partial z}, c\frac{\partial}{\partial y} - b\frac{\partial}{\partial z}$$
and we have
$$\begin{aligned}
[-b\frac{\partial}{\partial x} + a\frac{\partial}{\partial y}, c\frac{\partial}{\partial x} - a\frac{\partial}{\partial z}] &= [-b\frac{\partial}{\partial x}, -a\frac{\partial}{\partial z}] + [a\frac{\partial}{\partial y}, c\frac{\partial}{\partial x}] + [a\frac{\partial}{\partial y}, -a\frac{\partial}{\partial z}] \\
&= b\frac{\partial a}{\partial x}\frac{\partial}{\partial z} - a\frac{\partial b}{\partial z}\frac{\partial}{\partial x} + a\frac{\partial c}{\partial y}\frac{\partial}{\partial x} - c\frac{\partial a}{\partial x}\frac{\partial}{\partial y} + -a\frac{\partial a}{\partial y}\frac{\partial}{\partial z} + a\frac{\partial a}{\partial z}\frac{\partial}{\partial y} \\
&= a\big((\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z})\frac{\partial}{\partial x} + \frac{\partial a}{\partial z}\frac{\partial}{\partial y} - \frac{\partial a}{\partial y}\frac{\partial}{\partial z}) + b\frac{\partial a}{\partial x}\frac{\partial}{\partial z} + -c\frac{\partial a}{\partial x}\frac{\partial}{\partial y}.
\end{aligned}$$
Since we are assuming that $\mathcal{F}$ is involutive we have
$$a\big((\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z})a + \frac{\partial a}{\partial z}b - \frac{\partial a}{\partial y}c) = 0.$$
Since $a \neq 0$ we have:
$$\big((\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z})a + \frac{\partial a}{\partial z}b - \frac{\partial a}{\partial y}c) = 0.$$
This same equation hold for any cyclic permutation of a, b, c and simultaneous permutation of x, y, z. Adding the resulting three equations gives
$$2\big((\frac{\partial c}{\partial y} - \frac{\partial b}{\partial z})a + (\frac{\partial a}{\partial z} - \frac{\partial c}{\partial x})b + (\frac{\partial b}{\partial x} - \frac{\partial a}{\partial y}c)\big) = 0.$$
as required. □

23 The holonomy of closed loop in a leaf

Definition 23.1. Let $\mathcal{F}$ be a foliation of a manifold M. A transversal to $\mathcal{F}$ is smooth locally closed submanifold of M which meets all leaves transversally. A local transversal is a transversal which is diffeomorphic to a disk.

To discuss the holonomy we will use the terminology of a germs.

Definition 23.2. Let X, Y be smooth manifolds. Fix a point $x \in X$. A germ of smooth mappings at x is the equivalence class of functions $f : U \to Y$ where $U \subset X$ is an open neighborhood of x under the equivalence relation of agreement upon restriction. That is $f : U \to Y$ is equivalent to $g : V \to Y$ if there is a neighborhood W of x so that $f|_W = g|_W$.

Let τ_1 and τ_2 be local transversals hitting the same leaf $\mathcal{L}$ of $\mathcal{F}$. τ_1 and τ_2 are both contained in the same foliation chart U. Then the chart defines the germ of a diffeomorphism from τ_1 at $\tau_1 \cap \mathcal{L}$ to τ_2 at $\tau_2 \cap \mathcal{L}$

Let $\gamma : S^1 \to \mathcal{L}$ be a C^1 closed loop based at x in a leaf $\mathcal{L}$ of foliation $\mathcal{F}$. Let τ be a transversal to $\mathcal{F}$ passing through x.

24 Reeb's stability theorem

Definition 24.1. A codimension one foliation is called transversally orientable if the normal bundle $\nu = TM/T\mathcal{F}$ is orientable.

Theorem 24.2. *Let $\mathcal{F}$ be a normally oriented two dimensional foliation of a compact oriented three manifold. If $\mathcal{F}$ contains S^2 as a closed leave then the pair $M, \mathcal{F}$ is diffeomorphic to $S^2 \times S^1$ with the product foliation by two-spheres..*

Remark 7. To see that the normally oriented condition is important in the statement of the result note the following. $S^2 \times S^1$ has an orientation preserving involution $\tau : S^2 \times S^1 \to S^2 \times S^1$ given by

$$\tau(x, e^{i\theta}) = (-x, e^{-i\theta}).$$

This is a fixed point free involution so the quotient $X = S^2 \times S^1/(x, e^{i\theta}) \sim (-x, e^{-i\theta})$ has the structure of manifold as well. The product foliation is of $S^2 \times S^1$ is carried to itself by τ and descends to a foliation of X. The induced foliation is not normally oriented (can you see this). Most of the leaves are two sphere but there are two leaves which are real projective planes.

Lemma 24.3. *Let $\phi : D^2 \to M$ be an smooth embedding of D^2 into M^3 with image contained in a leaf L of $\mathcal{F}$. Then there is a foliating coordinate patch $\tilde{\phi} : D^2 \times (-\epsilon, \epsilon) \to M^3$ extending ϕ.*

Proof. First of all it is straightforward to construct a coordinate patch $\psi : D^2 \times (-a, a) \to M$ extending ϕ so that $\mathcal{F}$ is transverse to all the $\psi(\{x\} \times (-a, a))$ and so $T\mathcal{F}$ agrees with $D_{(0,t)}\psi(T_0 D^2 \times \{0\})$. Transfer $\mathcal{F}$ to a foliation of $D^2 \times (-a, a)$ still called $\mathcal{F}$. Let (r, θ) be polar coordinates in the disk.

Define G on $(D^2 \setminus \{0\}) \times (-a, a)$ to be the span of $\frac{\partial}{\partial r}$ and $\frac{\partial}{\partial t}$. By construction G is transverse to $\mathcal{F}$ and so the intersection $T\mathcal{F} \cap G$ defines a line field on $(D^2 \setminus \{0\}) \times (-a, a)$. This line field is spanned by a vector field of the form $v(r, \theta, t) = \frac{\partial}{\partial r} + a(r, \theta, t)\frac{\partial}{\partial t}$. We have $a(r, \theta, 0) = 0$ and $a(0, \theta, t) = 0$. and let F_s denote the time s flow of v. $F_s(r, \theta, t) = (r + s, \theta, T_s(r, \theta, t))$ when it is defined. Choose b small enough so that the time 1-flow of v with initial conditions $(0, \theta, t)$ for $|t| < b$ is defined. Define a map $\tilde{\phi} : D^2 \times (-b, b) \to D^2 \setminus \{0\}) \times (-a, a)$ by sending (r, θ, t) to the point $(r, \theta, T_r(0, \theta, t))$ or in words the time r flow of $(0, \theta, t)$ under v. This map takes the line segment $\{(r, \theta, t)|0 \le r < 1\}$ to a leaf. Since for any θ $v(0, \theta, t) = \frac{\partial}{\partial r}$ is tangent to $\mathcal{F}$, $\tilde{\phi}$ carries $D^2 \times \{t\}$ onto a leaf. Thus $\tilde{\phi}$ is the required map. □

Next we prove that in a neighborhood of a two-sphere leaf the foliation has a product structure.

Lemma 24.4. *Suppose that $\mathcal{L}$ is a leaf of $\mathcal{F}$ which diffeomorphic to S^2 The is a saturated neighborhood N of $\mathcal{L}$ which diffeomorphic to $S^2 \times (-a, a)$ with the product foliation.*

Proof. Decompose $S^2 = D^2_+ \cup D^2_-$. By the previous lemma we can find standard neighborhoods and glue them together to get the result. □

Next we will show that the set of points on a leaf diffeomorphic to S^2 is both open and closed.

Theorem 24.5. *Let $\mathcal{F}$ be a transversally oriented foliation. Then there is a embedding $\gamma : S^1 \to M$ transverse to the leaves. In fact γ can be chosen to pass through any point of M*

Remark 8. This is not to say that the image of γ hits all the leaves. This is a much stronger condition. A foliation with this addition property is called taut. The Reeb foliation of S^3 is an example of a non-taut foliation. Any flow line can only touch the torus leaf once but a closed circle transverse to a torus in S^3 must meet the torus in an even number of points.

Proof. Fix a point $x_0 \in M$. Since $\mathcal{F}$ is transversally oriented there is a nowhere vanishing vector field, v, which is transverse to the leaves. Let F_t denote the time-t flow for this vector field and consider a particular flow line, γ, of this vector field. If this flow line is a periodic orbit we are done so suppose it is not. Then we claim that there is leaf that is hit infinitely often by the flowline. We can find $x \in X$ and sequence $t_i \to \infty$ so that $\lim_{i\to\infty} F_{t_i}(x_0) = x$. Let U be a foliation chart in M about x. We can construct a smaller chart, V, about x by using the vector field v to flow away from the leaf $\mathcal{L}$ containing x. In V if a point is on a connected component of the part of the flow line in V it hits $\mathcal{L}$. Since infinitely many points of γ in different components of $\gamma \cap V$ are contained in V the claim follows.

Thus we can find a piece of orbit which contains x_0 and hits some leaf twice and the points of intersection are contained in the patch V. It is straightforward to modify the piece of flow line in this patch to close it up. $\square$

Now consider our transversally oriented foliation of M^3 containing a leaf $\mathcal{L}$ diffeomorphic to S^2. Let γ be a closed transverse curve passing through $\mathcal{L}$. Let Γ denote the union of all the leaves which pass through Γ. We claim that Γ is all of M and that γ hits each leaf the same number of times.

By Lemma 24.4 Γ is open. Also by this lemma there for each point y of γ there is a compact foliated neighborhood diffeomorphic to $S^2 \times [0, 1]$. By the compactness of γ finitely many such neighborhoods cover γ but then Γ is the union of finitely many closed sets and hence closed. Finally consider the function which associates to each point y of γ the of points of γ contained in the same leaf as y. By Lemma 24.4 this is a continuous function and hence is constant.

Finally choose a new γ which hits $\mathcal{L}$ once and hence all leaves once. Then

$$h : \mathcal{L} \times \gamma \to M$$

given by taking $y \in \mathcal{L}$ and $t \in \gamma$ to the unique point in the leaf through t hit by the flow line of v through y is the required diffeomorphism.