18.950 Differential Geometry
 Fall 2008

CHAPTER 2

# Local geometry of hypersurfaces

# Lecture 11

Background from linear algebra: A symmetric bilinear form on $\mathbb{R}^n$ is a map $I : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ of the form $I(x,y) = \sum_{ij} x_i a_{ij} y_j$, where $a_{ij} = a_{ji}$. Equivalently, $I(x,y) = \langle x, Ay \rangle$, where $A$ is a symmetric matrix. We say that $I$ is an inner product if $I(x,x) > 0$ for all nonzero $x$, or equivalently if $A$ is positive definite.

Suppose from now on that $I$ is an inner product. A basis $(e_1, \ldots, e_n)$ is called orthogonal with respect to $I$ if

$$I(e_i, e_i) = 1, \quad I(e_i, e_j) = 0 \text{ for } i \neq j.$$

Such bases always exist. In particular, by passing from the standard basis to the basis given by such vectors, one reduces standard about $I$ to ones about the standard inner product $\langle \cdot, \cdot \rangle$. A linear map $L : \mathbb{R}^n \to \mathbb{R}^n$ is called selfadjoint with respect to $I$ if $I(x, Ly)$ is a symmetric bilinear form. Equivalently, this is the case iff $AL$ is symmetric, which means that

$$AL = L^{tr} A.$$

Such a matrix $L$ always has a basis of eigenvectors, which is an orthogonal basis with respect to $I$.

Background from multivariable calculus: the derivative or Jacobian of a smooth map $f : \mathbb{R}^m \to \mathbb{R}^n$ at a point $x$ is a linear map $Df_x : \mathbb{R}^m \to \mathbb{R}^n$. In terms of partial derivatives,

$$Df_x(X) = \left(\sum_j \partial_{x_j} f_1 \cdot X_j, \sum_j \partial_{x_j} f_2 \cdot X_j, \ldots\right).$$

The chain rule is $D(f \circ g)_x = Df_{g(x)} \cdot Dg_x$, where the right hand side is matrix multiplication. The second derivative is a symmetric bilinear map $D^2 f_x : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}^n$ (for $n = 1$, this is a symmetric bilinear form, called the Hessian of the function $f$). Again explicitly,

$$D^2 f_x(X, Y) = \left(\sum_{i,j} \partial^2_{x_i x_j} f_1 \cdot X_i Y_j, \sum_{i,j} \partial^2_{x_i x_j} f_2 \cdot X_i Y_j, \ldots\right).$$

# Lecture 12

DEFINITION 12.1. A *hypersurface patch* is a smooth map $f : U \to \mathbb{R}^{n+1}$, where $U \subset \mathbb{R}^n$ is an open subset, such that the derivatives $\partial_{x_1} f, \ldots, \partial_{x_n} f \in \mathbb{R}^{n+1}$ are linearly independent at each point $x$. Equivalently, the Jacobian $Df_x : \mathbb{R}^n \to \mathbb{R}^{n+1}$ is injective (one-to-one).

DEFINITION 12.2. Let $f$ be a hypersurface patch. There is a unique $\nu : U \to \mathbb{R}^{n+1}$ such that $\nu(x)$ is of length one, is orthogonal to $\partial_{x_1} f, \ldots, \partial_{x_n} f$, and satisfies $\det(\partial_{x_1} f, \ldots, \partial_{x_n} f, \nu(x)) > 0$. It is automatically smooth. We call $\nu(x)$ the *Gauss normal vector* of $f$ at the point $x$.

Like in Frenet theory, we have an explicit formula. First, define $N$ by

$$N_i = \det(\partial_{x_1} f, \ldots, \partial_{x_n} f, \overbrace{(0, \ldots, 1, \ldots, 0)}^{i\text{-th unit vector}}).$$

Then $\nu = N/\|N\|$. For a curve in $\mathbb{R}^2$, this simplifies to $\nu = Jf'/\|f'\|$. For a surface in $\mathbb{R}^3$, $N = \partial_{x_1} f \times \partial_{x_2} f$, hence

$$\nu = \partial_{x_1} f \times \partial_{x_2} f / \|\partial_{x_1} f \times \partial_{x_2} f\|.$$

DEFINITION 12.3. Define $G_{ij}(x) = \langle \partial_{x_i} f, \partial_{x_j} f \rangle$. Equivalently, the matrix with entries $G_{ij}(x)$ is $G_x = Df_x^{tr} \cdot Df_x$. The associated inner product, $I_x(X, Y) = \langle X, G_x Y \rangle = \langle Df_x(X), Df_x(Y) \rangle$, is called the *first fundamental form*.

DEFINITION 12.4. Define $H_{ij}(x) = -\langle \partial_{x_i} \nu, \partial_{x_j} f \rangle = \langle \partial_{x_i} \partial_{x_j} f, \nu(x) \rangle$. Equivalently, the matrix with entries $H_{ij}(x)$ is $H_x = -D\nu_x^{tr} \cdot Df_x$. The associated symmetric bilinear form, $II_x(X, Y) = \langle X, H_x Y \rangle = -\langle D\nu_x(X), Df_x(Y) \rangle = \langle \nu(x), D^2 f_x(X, Y) \rangle$, is called the *second fundamental form*.

DEFINITION 12.5. Define a matrix $L_x$ by $L_x = G_x^{-1} H_x = (H_x G_x^{-1})^{tr}$. We call this the *shape operator*. Equivalently, this is characterized by the property that

$$II_x(X, Y) = I_x(L_x X, Y).$$

LEMMA 12.6. $D\nu = -Df \cdot L$ (matrix multiplication). More explicitly, each partial derivative $\partial_{x_i} \nu$ lies in the linear span of $\{\partial_{x_1} f, \ldots, \partial_{x_n} f\}$, and the shape operator allows us to express it as a linear combination of these vectors:

$$\partial_{x_i} \nu = -\sum_j L_{ji}(x) \partial_{x_j} f.$$

EXAMPLE 12.7. Suppose that $f(x) = (x, h(x))$, where $h$ is a smooth function of $n$ variables. Let $p \in U$ be a point where $h$ and $Dh$ both vanish. At that point, $G = \mathbf{1}$ is the identity matrix, and $H$ (as well as $L$) is the Hessian $D^2 h$.

# Lecture 13

Here's a summary. Let $f : U \to \mathbb{R}^{n+1}$ be a hypersurface patch, and $\nu : U \to \mathbb{R}^{n+1}$ its Gauss map. We then get:

| coefficients | matrix | bilinear form |
|---|---|---|
| $G_{ij} = \langle \partial_{x_i} f, \partial_{x_j} f \rangle$ $\quad$ $H_{ij} = -\langle \partial_{x_i} \nu, \partial_{x_j} f \rangle$ $\quad = \langle \nu, \partial^2_{x_i x_j} f \rangle$ $\quad$ $L_{ij}$ | $G = Df^{tr} \cdot Df$ $\quad$ $H = -D\nu^{tr} \cdot Df$ $\quad$ $L = G^{-1}H = (HG^{-1})^{tr}$ | $I(X,Y) = \langle Df(X), Df(Y) \rangle$ $\quad$ $II(X,Y) = -\langle D\nu(X), Df(Y) \rangle$ $\quad = \langle \nu, D^2 f(X,Y) \rangle$ |

Let $U, \tilde{U}$ be open subsets of $\mathbb{R}^n$, and $\phi : \tilde{U} \to U$ a smooth map such that $\det(D\phi) > 0$ everywhere. If $f : U \to \mathbb{R}^{n+1}$ is a regular hypersurface, then so is $\tilde{f} = f \circ \phi : \tilde{U} \to \mathbb{R}^{n+1}$, which we call a *partial reparametrization* of $f$.

PROPOSITION 13.1. The coordinate changes for the main associated data are

$$\tilde{\nu}(x) = \nu(\phi(x)),$$

$$\tilde{G}(x) = D\phi(x)^{tr} \cdot G(\phi(x)) \cdot D\phi(x),$$

$$\tilde{H}(x) = D\phi(x)^{tr} \cdot H(\phi(x)) \cdot D\phi(x),$$

$$\tilde{L}(x) = D\phi(x)^{-1} \cdot L(\phi(x)) \cdot D\phi(x).$$

All the structures above are obtained by differentiating $f$. It is interesting to ask to what extent they can be integrated back to determine the hypersurface itself.

EXAMPLE 13.2. Let $f : U \to \mathbb{R}^{n+1}$ be a hypersurface patch such that $L$ is $1/R$ times the identity matrix, for some $R \neq 0$. Then $f + R\nu$ is constant, and therefore, the image $f(U)$ is contained in a radius $|R|$ sphere in $\mathbb{R}^{n+1}$.

PROPOSITION 13.3. Let $f, \tilde{f} : U \to \mathbb{R}^{n+1}$ be two hypersurface patches, defined on the same *connected* set $U \subset \mathbb{R}^n$. Suppose that their first and second fundamental forms coincide. Then $\tilde{f}(x) = Af(x) + c$, where $A$ is an orthogonal matrix with determinant $+1$, and $c$ some constant.

# Lecture 14

By definition $L_x$ is selfadjoint with respect to the inner product $I_x$. Hence, it has a basis of eigenvectors which are orthonormal with respect to $I_x$. Note that $X$ is an eigenvector of $L_x$ iff $H_x X = \lambda G_x X$. Hence, the eigenvalues $\lambda$ are the solutions of $\det(G - \lambda H) = 0$.

DEFINITION 14.1. The eigenvalues $(\lambda_1, \ldots, \lambda_n)$ of $L_x$ are called the *principal curvatures* of the hypersurface patch $f$ at $x$. The corresponding eigenvectors $(X_1, \ldots, X_n)$ are called the *principal curvature directions*.

If $\tilde{f} = f(\phi)$ is a partial reparametrization of $f$, then the principal curvatures of $\tilde{f}$ at $x$ are equal to the principal curvatures of $f$ at $\phi(x)$.

EXAMPLE 14.2. Suppose that $f$ is such that $f_1$ achieves its maximum at the point $p$. Then $\nu(p) = (\pm 1, 0, \ldots, 0)$. In the $+$ case, all principal curvatures at $p$ are $\leq 0$. In the $-$ case, all principal curvatures at $p$ are $\geq 0$.

EXAMPLE 14.3. Suppose that $f$ is such that $\|f\|$ achieves its maximum at the point $p$, where $\|f(p)\| = R$. Then $\nu(p) = \pm f(p)/\|f(p)\|$. In the $+$ case, all principal curvatures at $p$ are $\leq -1/R < 0$. In the $-$ case, all principal curvatures at $p$ are $\geq 1/R > 0$.

DEFINITION 14.4. Let $\lambda_1, \ldots, \lambda_n$ be the principal curvatures of $f$ at $x$. The *mean curvature* is

$$\kappa_{mean} = \lambda_1 + \cdots + \lambda_n = \operatorname{trace}(L).$$

The *Gauss curvature* is

$$\kappa_{gauss} = \lambda_1 \cdots \lambda_n = \det(L) = \det(H)/\det(G).$$

The *scalar curvature* is

$$\kappa_{scalar} = \sum_{i<j} \lambda_i \lambda_j = \tfrac{1}{2}(\operatorname{trace}(L)^2 - \operatorname{trace}(L^2)).$$

LEMMA 14.5. The Gauss curvature is

$$\kappa_{gauss} = (-1)^n \frac{\det(\partial_{x_1}\nu, \ldots, \partial_{x_n}\nu, \nu)}{\sqrt{\det G}}.$$

## Lecture 15

EXAMPLE 15.1. Let $c$ be a Frenet curve in $\mathbb{R}^3$, parametrized with unit speed. Consider the surface patch $f(x_1, x_2) = c(x_1) + x_2 c'(x_1)$, where $x_2 > 0$. Then $\kappa_{gauss} = 0$ and

$$\kappa_{mean} = -\frac{1}{x_2} \cdot \frac{\tau(x_1)}{\kappa(x_1)},$$

where $\tau$ and $\kappa$ are the torsion and curvature of $c$ as a Frenet curve.

EXAMPLE 15.2. Let $c : I \to \mathbb{R}^2$ be a curve, parametrized with unit speed, whose first component $c_1$ is always positive. The associated surface of rotation is $f : I \times \mathbb{R} \to \mathbb{R}^3$,

$$f(x_1, x_2) = (c_1(x_1) \cos x_2, c_1(x_1) \sin x_2, c_2(x_1)).$$

The first and second fundamental forms of $f$ are given by

$$G = \begin{pmatrix} 1 & \\ & c_1^2 \end{pmatrix}, \quad H = \begin{pmatrix} -c_1'' c_2' + c_1' c_2'' & \\ & c_1 c_2' \end{pmatrix};$$

In particular, $\kappa_{gauss} = -c_1''/c_1$.

This can be used to construct surfaces with constant Gauss curvature, by solving the corresponding equation. For instance, the pseudo-sphere with Gauss curvature $-1$ is obtained by setting

$$c_1(t) = e^t, \quad c_2(t) = \int_0^t \sqrt{1 - e^{2\tau}} \, d\tau,$$

where $t \in (-\infty, 0)$.

**Lecture 16**

DEFINITION 16.1. Write
$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_k \Gamma_{ij}^k \frac{\partial f}{\partial x_k} + H_{ij}\nu.$$

The functions $\Gamma_{ij}^k(x)$ are called *Christoffel symbols*.

From the definition, it follows that
$$\sum_l \Gamma_{ij}^l G_{kl} = \langle \frac{\partial^2 f}{\partial x_i \partial x_j}, \frac{\partial f}{\partial x_k} \rangle.$$

THEOREM 16.2. Let $g^{ij}$ be the coefficients of the inverse matrix $G^{-1}$. Then
$$\Gamma_{ij}^l = \tfrac{1}{2} \sum_k g^{kl} \Big( \partial_{x_j} G_{ik} - \partial_{x_k} G_{ij} + \partial_{x_i} G_{jk} \Big).$$

The expression above shows that the Christoffel symbols only depend on the first fundamental form. By taking the definition of $\Gamma_{ij}^l$ and applying $\partial/\partial x_k$, we get
$$\sum_l \langle \frac{\partial^3 f}{\partial x_i \partial x_j \partial x_k}, \frac{\partial f}{\partial x_l} \rangle G^{ls} = \partial_k \Gamma_{ij}^s + \sum_t \Gamma_{ij}^t \Gamma_{kt}^s - H_{ij} L_{sk}.$$

Using cancellation properties on the left hand side, one sees that

THEOREM 16.3. The *Gauss equation* holds:
$$H_{ij} L_{sk} - H_{ik} L_{sj} = \partial_k \Gamma_{ij}^s - \partial_j \Gamma_{ik}^s + \sum_t \Gamma_{ij}^t \Gamma_{kt}^s - \Gamma_{ik}^t \Gamma_{jt}^s.$$

The expression on the right hand side of the Gauss equation is usually written as $R_{ikj}^s$. Denote by $\Gamma_i$ the matrices whose entries are the Christoffel symbols, more precisely
$$(\Gamma_j)_{si} = \Gamma_{ij}^s.$$

Similarly, write $R_{ij}$ for the matrices whose entries are the Riemann curvatures, more precisely
$$(R_{kj})_{si} = R_{ikj}^s.$$

Then, the definition of the $R_{ikj}^s$ can be rewritten in matrix notation as
$$R_{kj} = \partial_k \Gamma_j - \partial_j \Gamma_k + \Gamma_k \Gamma_j - \Gamma_j \Gamma_k.$$

# Lecture 17

Since $H = GL$, we can also write the Gauss equation in one of the two following forms:

$$H_{ij}H_{sk} - H_{ik}H_{sj} = \sum_u G_{su}R^u_{ikj},$$

$$L_{ij}L_{sk} - L_{ik}L_{sj} = \sum_u G^{iu}R^s_{ukj}.$$

For a surface in $\mathbb{R}^3$, one sets $(i, j, k, s) = (1, 1, 2, 2)$ in the first equation to get $\det(H)$, hence:

COROLLARY 17.1. (Theorema egregium for surfaces) The Gauss curvature of a surface patch is given in terms of the first fundamental form by

$$\kappa_{gauss} = \frac{\sum_u G_{2u}R^u_{121}}{\det(G)}.$$

EXAMPLE 17.2 (Isothermal or conformal coordinates). Suppose that the first fundamental form satisfies

$$G(x_1, x_2) = e^{h(x_1, x_2)} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Then $\kappa_{gauss} = -\frac{1}{2e^h}\Delta h$, where $\Delta$ is the Laplace operator. There is a (hard) theorem which says that for an arbitrary surface patch and any given point, one can find a local reparametrization which brings the metric into this form.

EXAMPLE 17.3 (Parallel geodesic coordinates). Suppose that the first fundamental form satisfies

$$G(x_1, x_2) = \begin{pmatrix} 1 & 0 \\ 0 & h^2(x_1, x_2) \end{pmatrix}.$$

Then $\kappa_{gauss} = -\frac{\partial^2_{x_1} h}{h}$. There is a (not so hard)which says that for an arbitrary surface patch and any given point, one can find a local reparametrization which brings the metric into this form.

# Lecture 18

We now introduce a generalization of our usual formalism, where the partial derivatives $\partial_{x_i} f$ are replaced by some more flexible auxiliary choice of basis at any point.

DEFINITION 18.1. Let $f : U \to \mathbb{R}^{n+1}$ be a hypersurface patch. A *moving basis* for $f$ is a collection $(X_1, \ldots, X_n)$ of vector-valued functions $X_i : U \to \mathbb{R}^n$ which are linearly independent at each point. If the $X_i$ are orthonormal with respect to the first fundamental form, we call $(X_1, \ldots, X_n)$ a *moving frame*.

Let $X$ be the matrix whose columns are $(X_1, \ldots, X_n)$, and define the *connection matrices* and their *curvature matrices* to be, respectively,

$$A_j = X^{-1}(\partial_{x_j} X) + X^{-1} \Gamma_j X,$$
$$F_{kj} = \partial_k A_j - \partial_j A_k + A_k A_j - A_j A_k.$$

LEMMA 18.2. For any moving basis, $F_{kj} = X^{-1} R_{kj} X$.

LEMMA 18.3. If the moving basis is a frame, the $A_j$ and $F_{kj}$ are skew-symmetric matrices.

Let's specialize to the case of surfaces, $n = 2$, and take $X$ to be a moving frame. Then, $F_{12}$ is necessarily a multiple of $J$. From the Gauss equation, we have

$$\begin{aligned}
\kappa_{gauss} &= \det(L) = (R_{21} G^{-1})_{12} \\
&= (X F_{21} X^{-1} G^{-1})_{12} = (X F_{21} X^{tr})_{12} \\
&= (F_{21})_{12} \det(X) = (F_{21})_{12} \det(G)^{-1/2}.
\end{aligned}$$

This gives rise to a curvature expression in curl form:

PROPOSITION 18.4. If $\alpha_i = (A_i)_{12}$, then

$$\kappa_{gauss} \sqrt{\det(G)} = (F_{21})_{12} = \partial_2 \alpha_1 - \partial_1 \alpha_2.$$

COROLLARY 18.5 (Gauss-Bonnet for tori). Let $f : \mathbb{R}^2 \to \mathbb{R}^3$ be a doubly-periodic surface patch, which means that $f(x_1 + T_1, x_2) = f(x_1, x_2) = f(x_1, x_2 + T_2)$ for some $T_1, T_2 > 0$. Then

$$\kappa_{gauss}^{tot} \overset{\text{def}}{=} \int_{[0,T_1] \times [0,T_2]} \kappa_{gauss} \sqrt{\det(G)} \, dx_1 dx_2 = 0.$$

From this and Example 14.3, we get:

COROLLARY 18.6. If $f$ is a doubly-periodic surface patch, then the Gauss curvature must be $> 0$ at some point, and $< 0$ at some other point.

**Lecture 19**

Before continuing, we need more linear algebra preliminaries: write $\Lambda^2(\mathbb{R}^n)$ for the space of skewsymmetric matrices of size $n$. This is a linear subspace of $\mathbb{R}^{n^2}$ of dimension $n(n-1)/2$. Given $v, w \in \mathbb{R}^n$, we denote by $v \wedge w$ the skewsymmetric matrix with entries

$$(v \wedge w)_{ij} = \tfrac{1}{2}(v_i w_j - w_i v_j).$$

This satisfies the rules

$$w \wedge v = -v \wedge w,$$
$$w \wedge (u + v) = w \wedge u + w \wedge v.$$

LEMMA 19.1. If $(v_i)_{1 \le i \le n}$ is any basis of $\mathbb{R}^n$, then $(v_i \wedge v_j)_{1 \le i < j \le n}$ is a basis of the space of antisymmetric matrices.

Given any linear map $L : \mathbb{R}^n \to \mathbb{R}^n$, there is an associated map

$$\Lambda^2 L : \Lambda^2 \mathbb{R}^n \longrightarrow \Lambda^2 \mathbb{R}^n, \quad (\Lambda^2 L)(S) = LSL^{tr}.$$

This satisfies (and is characterized by)

$$\Lambda^2 L(v \wedge w) = Lv \wedge Lw.$$

EXAMPLE 19.2. If $n = 2$, then $\Lambda^2 \mathbb{R}^2$ is one-dimensional, and $\Lambda^2 L$ is just multiplication with $\det(L)$.

LEMMA 19.3. We have

$$\text{trace}(\Lambda^2 L) = \tfrac{1}{2}(\text{trace}(L)^2 - \text{trace}(L^2)),$$
$$\det(\Lambda^2 L) = \det(L)^{n-1}.$$

LEMMA 19.4. Suppose that $L, \tilde{L} : \mathbb{R}^n \to \mathbb{R}^n$ are two linear maps, with $\text{rank}(L) \ge 3$. Then, if $\Lambda^2 L = \Lambda^2 \tilde{L}$, it also follows that $L = \pm \tilde{L}$.

This is easiest to see if $L$ is a diagonal matrix with entries $(1, \ldots, 1, 0, \ldots, 0)$, and the general case follows from that.

# Lecture 20

An expression is called *intrinsic* if it depends only on the first fundamental form and its derivatives. For instance, $G$ is intrinsic, but $H$ is not intrinsic. Less obviously, the Christoffel symbols are intrinsic, and so are the $R^s_{ikj}$. The last-mentioned observation deserves to be formulated in a more conceptual way.

Let $\Lambda^2 L : \Lambda^2 \mathbb{R}^n \to \Lambda^2 \mathbb{R}^n$ be the second exterior product of the shape operator. We call this the *Riemann curvature operator*, and denote it by $\mathcal{R}$. By definition

$$\mathcal{R}(e_j \wedge e_k) = Le_j \wedge Le_k = \sum_{is} L_{ij} L_{sk} e_i \wedge e_s =$$

$$\sum_{i<s} (L_{ij} L_{sk} - L_{sj} L_{ik}) e_i \wedge e_s = \sum_{i<s} \Big( \sum_u g^{iu} R^s_{ukj} \Big) e_i \wedge e_s.$$

Under reparametrization $\tilde{f} = f \circ \phi$, the Riemann curvature operators satisfy

$$\tilde{\mathcal{R}}(x) = (\Lambda^2 D\psi(x))^{-1} \cdot \mathcal{R}(\psi(x)) \cdot (\Lambda^2 D\psi(x)).$$

THEOREM 20.1. (Generalized theorema egregium) $\mathcal{R}$ is intrinsic.

COROLLARY 20.2. The unordered collection of $n(n-1)/2$ numbers $\lambda_i \lambda_j$ is intrinsic.

COROLLARY 20.3. $\kappa_{scalar}$ and $\kappa_{gauss}^{n-1}$ are intrinsic. In particular, $\kappa_{gauss}$ is intrinsic for $n$ even, and $|\kappa_{gauss}|$ is intrinsic for $n \geq 3$ odd.

COROLLARY 20.4. Let $f : U \to \mathbb{R}^{n+1}$ be a hypersurface patch, defined on a connected set. Suppose that for each point in $U$, the matrix $H_x$ has rank $\geq 3$. In that case, the intrinsic geometry of $f$ determines the extrinsic one. This means that if $\tilde{f} : U \to \mathbb{R}^{n+1}$ is another hypersurface patch with the same first fundamental form as $f$, then necessarily $\tilde{f}(x) = Af(x) + c$ with $A$ an orthogonal matrix, and $c$ a constant.

**Lecture 21**

To get some intuition for the intrinsic viewpoint, let's look at the problem of simplifying the first fundamental form by a local change of coordinates. More precisely, let $f : U \to \mathbb{R}^{n+1}$ be a hypersurface patch, and $p$ a point of $U$. A *local reparametrization* near $p$ is a partial reparametrization $\tilde{f} = f \circ \phi : \tilde{U} \to \mathbb{R}^{n+1}$, where $p \in \tilde{U}$ and $\psi(p) = p$. Such local reparametrizations are easy to find, because $\det(D\phi(p)) > 0$ implies positivity of that determinant for points close to $p$.

LEMMA 21.1. For any point $p$, there is always a local reparametrization such that in the new coordinates, $\tilde{G}_p = \mathbf{1}$ is the identity matrix.

LEMMA 21.2. Suppose that we have numbers $S_{ijk}$ (the indices $i,j,k$ run from 1 to $n$) such that $S_{ijk} = S_{jik}$. Then there are numbers $T_{ijk}$ with $T_{ijk} = T_{kji}$ such that

$$S_{ijk} = T_{ijk} + T_{jik}.$$

COROLLARY 21.3. For any point $p$, there is always a local reparametrization such that in the new coordinates, $\tilde{G}_p = \mathbf{1}$ and $\partial_{x_k} \tilde{G}_p = 0$ for all $k$.

## Lecture 22

Our first generalization is to hypersurfaces in Minkowski space. Take $\mathbb{R}^{n+1}$ with the Minkowski form $\langle X, Y \rangle_{Min} = X_1 Y_1 + X_2 Y_2 + \cdots + X_n Y_n - X_{n+1} Y_{n+1}$.

DEFINITION 22.1. A *spacelike hypersurface* in Minkowski space is a smooth map $f : U \to \mathbb{R}^{n+1}$, where $U \subset \mathbb{R}^n$ is an open subset, such that at every point $x \in U$, the derivatives $(\partial_{x_1} f, \ldots, \partial_{x_n} f)$ are linearly independent and span a subspace of $\mathbb{R}^{n+1}$ on which $\langle \cdot, \cdot \rangle_{Min}$ is positive definite.

More concretely, $f$ is spacelike if the matrices $G(x)$ with entries $G_{ij}(x) = \langle \partial_{x_i} f, \partial_{x_j} f \rangle_{Min}$ are positive definite for all $x$. We define this to be the first fundamental form of the hypersurface. Using the usual intrinsic formulae, we can now define the Christoffel symbols $\Gamma_{ij}^k$ and the $R_{ujk}^s$, hence the Riemann curvature operator $\mathcal{R}$.

DEFINITION 22.2. The Gauss normal vector of a spacelike hypersurface is the unique $\nu = \nu(x)$ such that $\langle \nu, \nu \rangle_{Min} = -1$, $\langle \nu, \partial_{x_i} f \rangle_{Min} = 0$, and $\det(\partial_{x_1} f, \ldots, \partial_{x_n} f, \nu) > 0$.

Given that, we now define $H$ by $H_{ij} = -\langle \partial_{x_i} \nu, \partial_{x_j} f \rangle_{Min} = \langle \nu, \partial_{x_i x_j}^2 f \rangle_{Min}$ and $L = G^{-1} H$. Some of the usual equations pick up additional signs, for instance:
$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_k \Gamma_{ij}^k \, \partial_{x_k} f - H_{ij} \nu.$$

Similarly, the theorema egregium says that $\mathcal{R} = -\Lambda^2(L)$. In particular, for spacelike surfaces, the Gauss curvature is $\kappa_{gauss} = -\det(H)/\det(G) = -\det(L)$.

LEMMA 22.3 (no proof). If $X \in \mathbb{R}^{n+1}$ has $\langle X, X \rangle_{Min} < 0$, then its Minkowski orthogonal complement $X^{\perp} = \{ Y \in \mathbb{R}^{n+1} : \langle X, Y \rangle_{Min} = 0 \}$ has the property that $\langle \cdot, \cdot \rangle_{Min}$ restricted to $X^{\perp}$ is positive definite.

EXAMPLE 22.4. *Hyperbolic $n$-space* is defined to be $H^n = \{ X \in \mathbb{R}^{n+1} : X_{n+1} > 0, \langle X, X \rangle_{Min} = -1 \}$. Suppose that $f : U \to \mathbb{R}^{n+1}$ is some parametrization of $H^n$. Since $\langle f, \partial_{x_i} f \rangle = 0$, it follows from the Lemma that $f$ is spacelike. It has Gauss normal vector $\nu = \pm f$. Hence $H = \mp G$ and $L = \mp \mathbf{1}$. Hence, $\kappa_{gauss} = -1$.

Two explicit parametrizations of hyperbolic $n$-space: the first is the *Poincaré* or *conformal ball* model
$$f : U = \{ x \in \mathbb{R}^n : \|x\| < 1 \} \longrightarrow \mathbb{R}^{n+1},$$
$$f(x) = \frac{1}{1 - \|x\|^2} (2x_1, \ldots, 2x_n, 1 + \|x\|^2).$$

Geometrically, this corresponds to taking a disc in $\mathbb{R}^n \times \{0\}$, and then projecting radially from the point $(0, \ldots, -1)$. In this model,

$$G_{ij} = \begin{cases} \frac{4}{(1-\|x\|^2)^2} & i = j, \\ 0 & i \neq j. \end{cases}$$

The second is the *Klein* or *projective ball* model

$$\tilde{f} : U = \{x \in \mathbb{R}^n \ : \ \|x\| < 1\} \longrightarrow \mathbb{R}^{n+1},$$

$$\tilde{f}(x) = \frac{1}{\sqrt{1-\|x\|^2}}(x_1, \ldots, x_n, 1).$$

Geometrically, one takes the disc tangent to $H^n$ at the point $(0, \ldots, 0, 1)$, and then projects radially from the origin. The resulting first fundamental form is

$$\tilde{G}_{ij} = \begin{cases} \frac{1}{1-\|x\|^2} + \frac{x_i^2}{(1-\|x\|^2)^2} & i = j, \\ \frac{x_i x_j}{(1-\|x\|^2)^2} & i \neq j. \end{cases}$$

# Lecture 23

Our second generalization is to submanifolds which are not hypersurfaces. Let $U \subset \mathbb{R}^n$ be an open subset. A regular map (or *immersion*) $f : U \to \mathbb{R}^{n+m}$ is a smooth map such that the partial derivatives $\partial_{x_1} f, \ldots, \partial_{x_n} f$ are linearly independent at each point. The first fundamental form is then defined as usual by

$$G = Df^{tr} \cdot Df.$$

DEFINITION 23.1. A set of Gauss normal vectors for $f$ consists of maps $\nu^1, \ldots, \nu^m : U \to \mathbb{R}^{n+m}$ satisfying

$$\langle \nu^w, \nu^w \rangle = 1,$$
$$\langle \nu^v, \nu^w \rangle = 0 \text{ for } u \neq w,$$
$$\langle \nu^w, \partial_{x_i} f \rangle = 0,$$
$$\det(\partial_{x_1} f, \ldots, \partial_{x_n} f, \nu^1, \ldots, \nu^m) > 0.$$

Such maps may not necessarily exist over all of $U$, but they can be defined locally near any given $x \in U$ by the Gram-Schmidt method. Moreover, any two choices defined on the same subset are related by

$$\tilde{\nu}^w = \sum_v a_{vw} \nu^v,$$

where $a_{vw}$ are the coefficients of an orthogonal matrix $A = A(x)$ with $\det(A) = 1$.

DEFINITION 23.2. Given a set of Gauss normal vectors, we define the second fundamental forms $H^w$, $w = 1, \ldots, m$, by

$$H_{ij}^w = -\langle \partial_i \nu^w, \partial_j f \rangle = \langle \nu^w, \partial^2 f / \partial x_i \partial x_j \rangle.$$

The corresponding shape operators are $L^w = G^{-1} H^w$.

One then has

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \sum_k \Gamma_{ij}^k \, \partial_{x_k} f + \sum_w H_{ij}^w \nu,$$

where the Christoffel symbols $\Gamma_{ij}^k$ are given by the usual intrinsic formulae. The Gauss equation says that

$$\sum_w H_{ij}^w L_{sk}^w - H_{ik}^w L_{sj}^w = \partial_k \Gamma_{ij}^s - \partial_j \Gamma_{ik}^s + \sum_t \Gamma_{ij}^t \Gamma_{kt}^s - \Gamma_{ik}^t \Gamma_{jt}^s.$$

It is easy to check explicitly that the left hand side is independent of the choice of $\nu^w$. The Riemann curvature operator, given by the usual intrinsic formulae, now reads

$$\mathcal{R} = \sum_w \Lambda^2(L^w).$$

Its eigenvalues are now less constrained than in the hypersurface case, hence the connection between intrinsic and extrinsic geometry is somewhat weaker.

Our final generalization is to a completely intrinsic viewpoint. A *Riemannian metric* on $U \subset \mathbb{R}^n$ is a family $G_x$ of positively definite symmetric $n \times n$ matrices, depending smoothly on $x \in U$. For any such metric, and independently of any embedding of $U$ into another space, one can define Christoffel symbols, the Riemann curvature operator, and all its dependent quantities (scalar curvature, for instance). The proof of Corollary 18.6, for instance, is purely intrinsic and shows the following:

COROLLARY 23.3. Take any Riemannian metric on $\mathbb{R}^2$ which is doubly-periodic, $G_{(x_1+T_1,x_2)} = G_{(x_1,x_2)} = G_{(x_1,x_2+T_2)}$. Then

$$\kappa_{gauss}^{tot} = \int_{[0,T_1] \times [0,T_2]} \kappa_{gauss} \sqrt{\det(G)} \, dx_1 dx_2 = 0.$$